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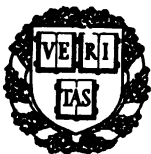
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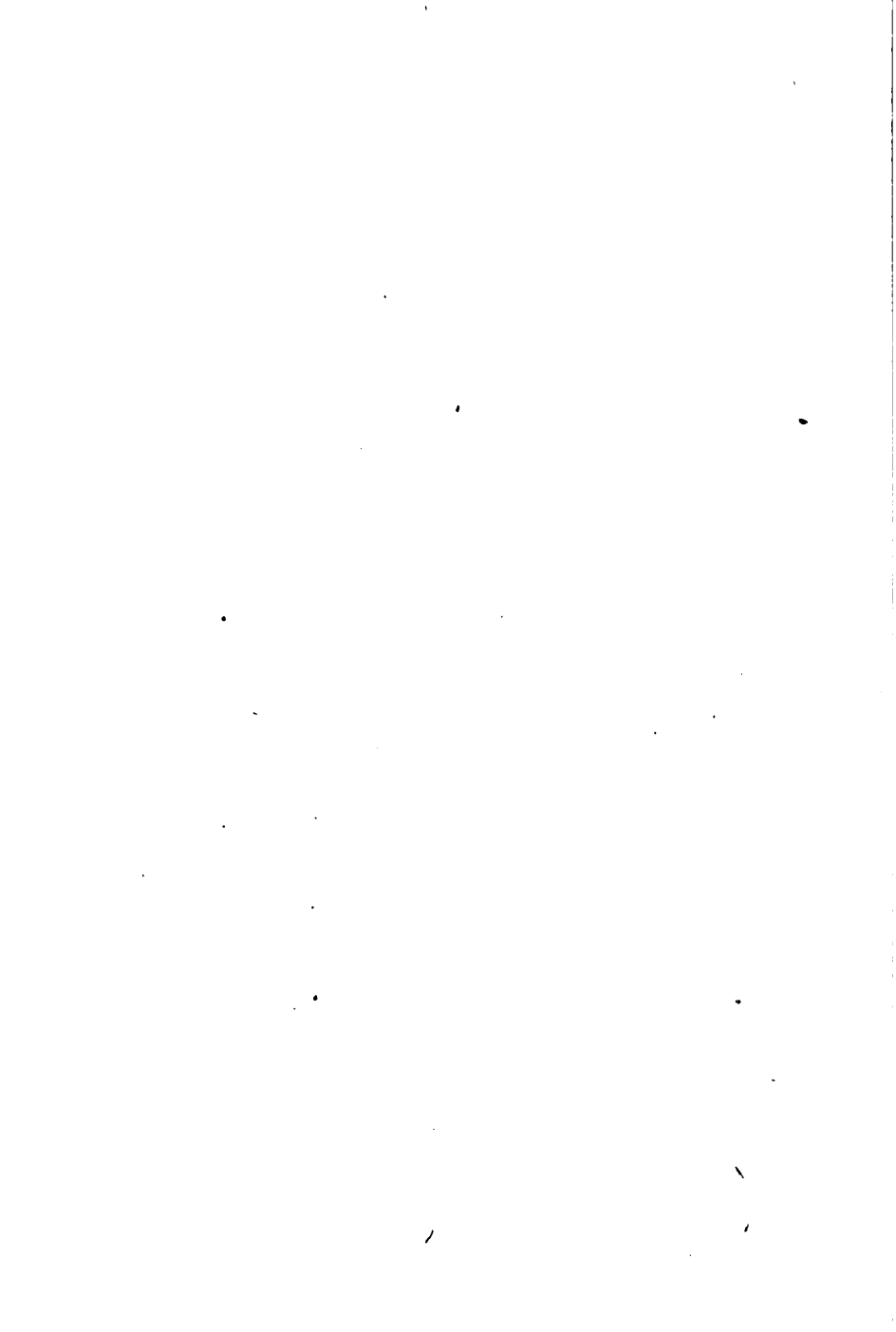
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PLANE
TRIGONOMETRY

BY

G. A. WENTWORTH

AUTHOR OF A SERIES OF TEXT-BOOKS IN MATHEMATICS

SECOND REVISED EDITION

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PREFACE

IN preparing this work the aim has been to furnish just so much of Trigonometry as is actually taught in our best schools and colleges. Consequently, all investigations that are important only for the special student have been omitted, except the development of functions in series. The principles have been unfolded with the utmost brevity consistent with simplicity and clearness, and interesting problems have been selected with a view to awaken a real love for the study. Much time and labor have been spent in devising the simplest proofs for the propositions, and in exhibiting the best methods of arranging the logarithmic work.

The author acknowledges his obligation to G. A. Hill, A.M., of Cambridge, Mass., to Dr. F. N. Cole, of New York, N.Y., to Professor S. F. Norris, of Baltimore, Md., and to Professor B. F. Yanney, of Alliance, Ohio.

G. A. WENTWORTH.

EXETER, N.H.,
January, 1903.



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PLANE TRIGONOMETRY

CHAPTER I

TRIGONOMETRIC FUNCTIONS OF ACUTE ANGLES

SECTION I

ANGULAR MEASURE

As lengths are measured in terms of various conventional units, the foot, the meter, etc., so different units for measuring angles are employed, or have been proposed.

In the **common** or **sexagesimal system** the circumference of a circle is divided into 360 equal parts. The angle at the centre subtended by each of these parts is taken as the unit angle and is called a **degree**. The degree is subdivided into 60 *minutes*, and the minute into 60 *seconds*. Degrees, minutes, and seconds are denoted by symbols. Thus, 6 degrees 5 minutes 7 seconds is written $6^{\circ} 5' 7''$.

NOTE. The sexagesimal system was employed by the early Babylonian astronomers to conform with their year of 360 days.

In the **circular system** an arc of a circle is laid off equal in length to the radius. The angle at the centre subtended by this arc is taken as the unit angle and is called a **radian**.

The number of radians in 360° is equal to the number of times the length of the radius is contained in the length of the circumference. It is proved in Geometry that this number is

2π for all circles, π being equal to 3.1416, nearly. Therefore the radian is the same angle in all circles.

The circumference of a circle is 2π times the radius.

Hence, 2π radians = 360° , and π radians = 180° .

Therefore, $1 \text{ radian} = \frac{180^\circ}{\pi} = 57^\circ 17' 45''$,

and $1 \text{ degree} = \frac{\pi}{180} \text{ radian} = 0.017453 \text{ radian}$.

By the last two equations the measure of an angle can be changed from radians to degrees or from degrees to radians.

Thus, $2 \text{ radians} = 2 \times \frac{180^\circ}{\pi} = 2 \times (57^\circ 17' 45'') = 114^\circ 35' 30''$.

NOTE. The circular system came into use early in the eighteenth century. It is found more convenient in the higher mathematics, where the radians are expressed simply as numbers. Thus, the angle π means π radians, and the angle 3 means 3 radians.

On the introduction of the metric system of weights and measures at the close of the eighteenth century, it was proposed to divide the right angle into 100 equal parts called *grades*, which were to be taken as units. The grade was subdivided into 100 *minutes* and the minute into 100 *seconds*. This *French* or *centesimal* system, however, never came into actual use.

EXERCISE I

[Assume $\pi = 3.1416$.]

1. Reduce the following angles to circular measure, expressing the results as fractions of π :

60° , 45° , 150° , 195° , $11^\circ 15'$, $123^\circ 45'$, $37^\circ 30'$.

2. How many degrees are there in $\frac{3}{4}\pi$ radians? $\frac{2}{3}\pi$ radians? $\frac{5}{6}\pi$ radians? $\frac{1}{3}\pi$ radians? $\frac{7}{8}\pi$ radians?

3. What decimal part of a radian is 1° ? $1'$?

4. How many seconds in a radian?

5. Express in radians one of the interior angles of a regular octagon; of a regular dodecagon.

6. On the circumference of a circle of 50 feet radius an arc of 10 feet is laid off. How many degrees in the angle at the centre subtended by this arc?

7. The earth's equatorial radius is approximately 3963 miles. If two points on the equator are 1000 miles apart, what is their difference in longitude?

8. If the difference in longitude of two points on the equator is 1° , what is the distance between them in miles?

9. What is the radius of a circle, if an arc of 1 foot subtends an angle of 1° at the centre?

10. In how many hours is a point on the equator carried by the rotation of the earth on its axis through a distance equal to the earth's radius?

11. The minute hand of a clock is $3\frac{1}{2}$ feet long. How far does its extremity move in 25 minutes? (Take $\pi = 3\frac{1}{2}$.)

12. A wheel makes 15 revolutions a second. How long does it take to turn through 4 radians? (Take $\pi = 3\frac{1}{2}$.)

SECTION II

THE TRIGONOMETRIC FUNCTIONS

The sides and angles of a plane triangle are so related that any three given parts, provided at least one of them is a side, determine the shape and the size of the triangle.

Geometry shows how, from three such parts, to **construct** the triangle.

Trigonometry shows how to **compute** the unknown parts of a triangle from the numerical values of the given parts.

Geometry shows in a general way that the sides and angles of a triangle are mutually dependent. Trigonometry begins

by showing the exact nature of this dependence in the **right triangle**, and for this purpose employs the **ratios of the sides**.

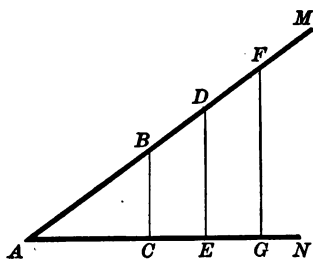


FIG. 1

Let MAN (Fig. 1) be an acute angle. If from any points B, D, F in one of its sides perpendiculars BC, DE, FG are let fall to the other side, then the right triangles ACB, AED, AGF thus formed have the angle A common, and are therefore mutually equiangular and similar. Hence, the ratios of their corresponding

sides, pair by pair, are equal. That is,

$$\frac{AC}{AB} = \frac{AE}{AD} = \frac{AG}{AF}; \quad \frac{AC}{BC} = \frac{AE}{DE} = \frac{AG}{FG}; \quad \frac{BC}{AB} = \frac{DE}{AD} = \frac{FG}{AF}.$$

These ratios, therefore, remain unchanged so long as the angle A remains unchanged.

Hence, for every value of an acute angle A there are certain *numbers* that express the values of the **ratios of the sides** in all right triangles that have this acute angle A .

There are all together six different ratios:

I. The ratio of the opposite leg to the hypotenuse is called the **Sine of A** , and is written **$\sin A$** .

II. The ratio of the adjacent leg to the hypotenuse is called the **Cosine of A** , and is written **$\cos A$** .

III. The ratio of the opposite leg to the adjacent leg is called the **Tangent of A** , and is written **$\tan A$** .

IV. The ratio of the adjacent leg to the opposite leg is called the **Cotangent of A** , and is written **$\cot A$** .

V. The ratio of the hypotenuse to the adjacent leg is called the **Secant of A** , and is written **$\sec A$** .

VI. The ratio of the hypotenuse to the opposite leg is called the **Cosecant of A** , and is written **$\csc A$** .

These six ratios are called the **Trigonometric Functions** of the angle A .

To these six ratios are often added the two following functions, which also depend only on the angle A :

VII. The **Versed Sine** of A is $1 - \cos A$, and is written **vers** A .

VIII. The **Coversed Sine** of A is $1 - \sin A$, and is written **covers** A .

In the right triangle ACB (Fig. 2) let a , b , c denote the lengths of the sides opposite the acute angles A , B , and the right angle C , respectively, these lengths being all expressed in terms of a common unit. Then,

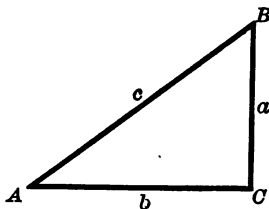


FIG. 2

$$\sin A = \frac{a}{c} = \frac{\text{opposite leg}}{\text{hypotenuse}},$$

$$\cos A = \frac{b}{c} = \frac{\text{adjacent leg}}{\text{hypotenuse}},$$

$$\tan A = \frac{a}{b} = \frac{\text{opposite leg}}{\text{adjacent leg}},$$

$$\cot A = \frac{b}{a} = \frac{\text{adjacent leg}}{\text{opposite leg}},$$

$$\sec A = \frac{c}{b} = \frac{\text{hypotenuse}}{\text{adjacent leg}},$$

$$\csc A = \frac{c}{a} = \frac{\text{hypotenuse}}{\text{opposite leg}},$$

$$\text{vers } A = 1 - \frac{b}{c} = \frac{c - b}{c},$$

$$\text{covers } A = 1 - \frac{a}{c} = \frac{c - a}{c}.$$

EXERCISE II

1. What are the functions of the other acute angle B of the triangle ACB (Fig. 2)?

2. Compare the functions of A and B , and show that

$$\sin A = \cos B,$$

$$\sec A = \csc B,$$

$$\cos A = \sin B,$$

$$\csc A = \sec B,$$

$$\tan A = \cot B,$$

$$\text{vers } A = \text{covers } B,$$

$$\cot A = \tan B,$$

$$\text{covers } A = \text{vers } B.$$

3. Find the values of the functions of A , if a, b, c , respectively, have the following values :

- (i) 3, 4, 5. (iii) 8, 15, 17. (v) 3.9, 8, 8.9.
 (ii) 5, 12, 13. (iv) 9, 40, 41. (vi) 1.19, 1.20, 1.69.

4. What condition must be fulfilled by the lengths of the three lines a, b, c (Fig. 2) in order to make them the sides of a right triangle? Is this condition fulfilled in Example 3?

5. Find the values of the functions of A , if a, b, c , respectively, have the following values :

- (i) $2mn, m^2 - n^2, m^2 + n^2$. (iii) pqr, qrs, rsp .
 (ii) $\frac{2xy}{x-y}, x+y, \frac{x^2+y^2}{x-y}$. (iv) $\frac{mn}{pq}, \frac{mv}{sq}, \frac{nr}{ps}$.

6. Prove that the values of a, b, c , in (i) and (ii), Example 5, satisfy the condition necessary to make them the sides of a right triangle.

7. What equations of condition must be satisfied by the values of a, b, c in (iii) and (iv), Example 5, in order that the values may represent the sides of a right triangle?

Given $a^2 + b^2 = c^2$; find the functions of A and B when:

8. $a = 24, b = 143$. 11. $a = \sqrt{p^2 + q^2}, b = \sqrt{2pq}$.
 9. $a = 0.264, c = 0.265$. 12. $a = \sqrt{p^2 + pq}, c = p + q$.
 10. $b = 9.5, c = 19.3$. 13. $b = 2\sqrt{pq}, c = p + q$.

Given $a^2 + b^2 = c^2$; find the functions of A when:

14. $a = 2b$. 16. $a + b = \frac{5}{4}c$.
 15. $a = \frac{3}{4}c$. 17. $a - b = \frac{1}{4}c$.
 18. Find a if $\sin A = \frac{3}{4}$, and $c = 20.5$.
 19. Find b if $\cos A = 0.44$, and $c = 3.5$.
 20. Find a if $\tan A = \frac{1}{3}$, and $b = 2\sqrt{11}$.
 21. Find b if $\cot A = 4$, and $a = 17$.

22. Find c if $\sec A = 2$, and $b = 20$.

23. Find c if $\csc A = 6.45$, and $a = 35.6$.

Construct a right triangle, given :

24. $c = 6$, $\tan A = \frac{3}{4}$.

26. $b = 2$, $\sin A = 0.6$.

25. $a = 3.5$, $\cos A = \frac{1}{2}$.

27. $b = 4$, $\csc A = 4$.

28. In a right triangle $c = 2.5$ miles, $\sin A = 0.6$, $\cos A = 0.8$; compute the legs.

29. Construct with a protractor the angles 20° , 40° , and 70° ; determine their functions by measuring the necessary lines, and compare the values obtained in this way with the more nearly correct values given in the following table :

	sin	cos	tan	cot	sec	csc
20°	0.342	0.940	0.364	2.747	1.064	2.924
40°	0.643	0.766	0.839	1.192	1.305	1.556
70°	0.940	0.342	2.747	0.364	2.924	1.064

30. Find, by means of the above table, the legs of a right triangle if $A = 20^\circ$, $c = 1$; also if $A = 20^\circ$, $c = 4$.

31. By dividing the length of a vertical rod by the length of its horizontal shadow, the tangent of the angle of elevation of the sun at the time of observation was found to be 0.82. How high is a tower, if the length of its horizontal shadow at the same time is 174.3 yards?

SECTION III

REPRESENTATION OF THE FUNCTIONS BY LINES

The functions of an angle, being ratios, are *numbers*; but we may represent them by *lines* if we first choose a unit of length, and then construct right triangles, such that the denominators of the ratios shall be equal to this unit.

The most convenient way is the following:

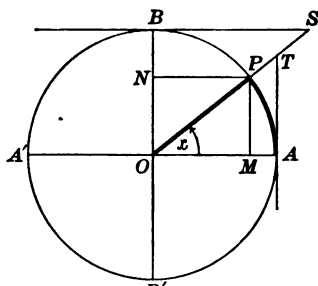


FIG. 3

About a point O (Fig. 3) a centre, with a radius equal one unit of length, describe circle, and draw the horizontal diameter AA' and the diameter BB' perpendicular to AA' .

A circle with radius equal to 1 is called a *unit circle*.

Let AOP be an acute angle and let its value (in degrees etc.) be denoted by x . We may

regard the angle x as generated by a line OP that revolves about O from the initial position OA to the terminal position OP .

Draw $PM \perp$ to OA , $PN \perp$ to OB .

In the rt. $\triangle OMP$ the hypotenuse $OP = 1$.

$$\text{Therefore, } \sin x = \frac{MP}{OP} = MP; \quad \cos x = \frac{OM}{OP} = OM.$$

Through A and B draw tangents to the circle meeting OP produced in T and S , respectively; then, in the rt. $\triangle OAT$ and OBS , $OA = 1$, the leg $OB = 1$, and the $\angle OSB =$ the $\angle x$. Therefore,

$$\tan x = \frac{AT}{OA} = AT; \quad \cot x = \frac{BS}{OB} = BS;$$

$$\sec x = \frac{OT}{OA} = OT; \quad \csc x = \frac{OS}{OB} = OS;$$

$$\text{vers } x = 1 - OM = MA; \quad \text{covers } x = 1 - ON = NB.$$

These eight *line values* of the functions are all expressed in terms of the radius of the circle as a unit; and it is clear that as the angle *varies in value* the *line values* of the functions will always remain equal *numerically* to the *ratio values*. Hence, in studying the changes in the functions as the angle

is supposed to vary in value, we may employ the simpler line values instead of the ratio values.

EXERCISE III

1. Represent by lines the functions of an acute angle larger than that shown in Fig. 3.

If x is an acute angle, show that:

2. $\sin x$ is less than $\tan x$.

3. $\sec x$ is greater than $\tan x$.

4. $\csc x$ is greater than $\cot x$.

Construct the angle x , if:

5. $\tan x = 3$. 7. $\cos x = \frac{1}{2}$. 9. $\sin x = 2 \cos x$.

6. $\csc x = 2$. 8. $\sin x = \cos x$. 10. $4 \sin x = \tan x$.

11. Show that the sine of an angle is equal to one-half the chord of twice the angle.

12. Find x if $\sin x$ is equal to one-half the side of a regular inscribed decagon.

Given x and y , $x + y$ being less than 90° ; construct:

13. The value of $\sin(x + y) - \sin x$.

14. The value of $\tan(x + y) - \sin(x + y) + \tan x - \sin x$.

Given an angle x ; construct an angle y such that:

15. $\sin y = 2 \sin x$.

17. $\tan y = 3 \tan x$.

16. $\cos y = \frac{1}{2} \cos x$.

18. $\sec y = \csc x$.

19. Show by construction that $2 \sin A > \sin 2A$.

20. Given two angles A and B , $A + B$ being less than 90° ; show that $\sin(A + B) < \sin A + \sin B$.

21. Given $\sin x$ in a unit circle; find the length of a line corresponding in position to $\sin x$ in a circle whose radius is r .

22. In a right triangle, given the hypotenuse c , and also $\sin A = m$, $\cos A = n$; find the legs.

SECTION IV

CHANGES IN THE FUNCTIONS AS THE ANGLE CHANGES

If we suppose the $\angle AOP$, or x (Fig. 4), to increase gradually to 90° by the revolution of the moving radius OP about O , the point P moves along the arc AB towards B , T moves along the tangent AT away from A , S moves along the tangent BS towards B , and M moves along the radius OA towards O .

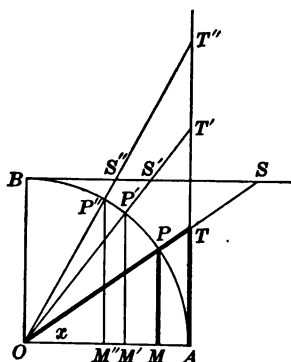


FIG. 4

Hence, the lines MP , AT , OT gradually increase in length, and the lines OM , BS , OS gradually decrease. That is,

As an acute angle increases to 90° , its sine, tangent, and secant also increase, while its cosine, cotangent, and cosecant decrease.

On the other hand, if we suppose x to decrease gradually, the reverse changes in its functions occur.

If we suppose x to decrease to 0° , OP coincides with OA and is parallel to BS . Therefore, MP and AT vanish, OM becomes equal to OA , while BS and OS are each infinitely long and are represented in value by the symbol ∞ .

And if we suppose x to increase to 90° , OP coincides with OB and is parallel to AT . Therefore, MP and OS are each equal to OB , OM and BS vanish, while AT and OT are each infinite in length.

Hence, as the angle x increases from 0° to 90° ,

$\sin x$ increases from 0 to 1,
 $\cos x$ decreases from 1 to 0,

$\tan x$ increases from 0 to ∞ ,
 $\cot x$ decreases from ∞ to 0,
 $\sec x$ increases from 1 to ∞ ,
 $\csc x$ decreases from ∞ to 1.

The values of the functions of 0° and of 90° are the *limiting* values of the functions of an acute angle. It is evident that for acute angles,

Sines and cosines are always less than 1;
 Secants and cosecants are always greater than 1;
 Tangents and cotangents have all values between 0 and ∞ .

REMARK. We are now able to understand why the sine, cosine, etc., of an angle are called *functions* of the angle. By a *function* of any magnitude is meant another magnitude which remains constant so long as the first magnitude remains constant, but changes in value for every change in the value of the first magnitude. This, as we now see, is the relation in which the sine, cosine, etc., of an angle stand to the angle.

SECTION V

FUNCTIONS OF COMPLEMENTARY ANGLES

The general form of two complementary angles is A and $90^\circ - A$.

In the rt. $\triangle ACB$ (Fig. 5),

$$A + B = 90^\circ; \text{ hence } B = 90^\circ - A.$$

Hence, putting $90^\circ - A$ for B in the formulas on p. 5,

$$\sin A = \cos B = \cos (90^\circ - A),$$

$$\cos A = \sin B = \sin (90^\circ - A),$$

$$\tan A = \cot B = \cot (90^\circ - A),$$

$$\cot A = \tan B = \tan (90^\circ - A),$$

$$\sec A = \csc B = \csc (90^\circ - A),$$

$$\csc A = \sec B = \sec (90^\circ - A).$$

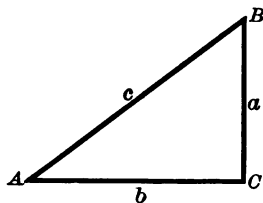


FIG. 5

Therefore,

Each function of an acute angle is equal to the co-named function of the complementary angle.

NOTE. Cosine, cotangent, and cosecant are sometimes called *co-functions*; the words are simply abbreviated forms of *complement's sine, complement's tangent, and complement's secant*.

Hence, also,

Any function of an angle between 45° and 90° may be found by taking the co-named function of the complementary angle between 0° and 45° .

EXERCISE IV

1. Express as functions of the complementary angle:

$\sin 30^\circ$	$\tan 89^\circ$	$\csc 18^\circ 10'$	$\cot 82^\circ 19'$
$\cos 45^\circ$	$\cot 15^\circ$	$\cos 37^\circ 24'$	$\csc 54^\circ 46'$

2. Express as functions of an angle less than 45° :

$\sin 60^\circ$	$\tan 57^\circ$	$\csc 69^\circ 2'$	$\cot 89^\circ 59'$
$\cos 75^\circ$	$\cot 84^\circ$	$\cos 85^\circ 39'$	$\csc 45^\circ 1'$

- Given $\tan 30^\circ = \frac{1}{\sqrt{3}}$; find $\cot 60^\circ$.
- Given $\tan A = \cot A$; find A .
- Given $\cos A = \sin 2A$; find A .
- Given $\sin A = \cos 2A$; find A .
- Given $\cos A = \sin (45^\circ - \frac{1}{2}A)$; find A .
- Given $\cot \frac{1}{2}A = \tan A$; find A .
- Given $\tan (45^\circ + A) = \cot A$; find A .
- Find A if $\sin A = \cos 4A$.
- Find A if $\cot A = \tan 8A$.
- Find A if $\cot A = \tan nA$.

SECTION VI

RELATIONS OF THE FUNCTIONS OF AN ANGLE

Since (Fig. 6) $a^2 + b^2 = c^2$, therefore,

$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = 1, \text{ or } \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1.$$

But $\frac{a}{c} = \sin A$, and $\frac{b}{c} = \cos A$;

therefore, $(\sin A)^2 + (\cos A)^2 = 1$;

or, as it is usually written for convenience,

$$\sin^2 A + \cos^2 A = 1. \quad [1]$$

That is: *The sum of the squares of the sine and the cosine of an angle is equal to unity.*

Formula [1] enables us to find the cosine of an angle when the sine is known, and the sine when the cosine is known.

The values of $\sin A$ and of $\cos A$ deduced from [1] are:

$$\sin A = \sqrt{1 - \cos^2 A}, \quad \cos A = \sqrt{1 - \sin^2 A}.$$

Since

$$\frac{a}{c} \div \frac{b}{c} = \frac{a}{c} \times \frac{c}{b} = \frac{a}{b},$$

and since $\frac{a}{c} = \sin A$, $\frac{b}{c} = \cos A$, and $\frac{a}{b} = \tan A$,

therefore, $\tan A = \frac{\sin A}{\cos A}. \quad [2]$

That is: *The tangent of an angle is equal to the sine divided by the cosine.*

Formula [2] enables us to find the tangent of an angle when the sine and the cosine are known.

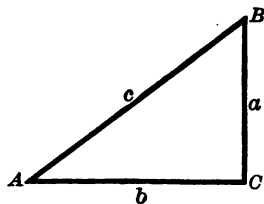


FIG. 6

Now

$$\frac{a}{c} \times \frac{c}{a} = 1, \quad \frac{b}{c} \times \frac{c}{b} = 1, \quad \frac{a}{b} \times \frac{b}{a} = 1,$$

and

$$\frac{a}{c} = \sin A, \quad \frac{b}{c} = \cos A, \quad \frac{a}{b} = \tan A,$$

$$\frac{b}{a} = \cot A, \quad \frac{c}{b} = \sec A, \quad \frac{c}{a} = \csc A.$$

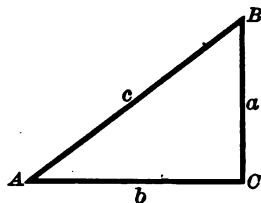


FIG. 7

Therefore,

$$\left. \begin{aligned} \sin A \times \csc A &= 1 \\ \cos A \times \sec A &= 1 \\ \tan A \times \cot A &= 1 \end{aligned} \right\} \quad [3]$$

That is: *The sine and the cosecant of an angle, the cosine and the secant, and the tangent and the cotangent, pair by pair, are reciprocals.*

The equations in [3] enable us to find an unknown function contained in any pair of these reciprocals when the other function in this pair is known.

EXERCISE V

1. Prove Formulas [1], [2], [3], using for the functions the line values in the unit circle given in Sect. III, page 8.

Prove that:

2. $1 + \tan^2 A = \sec^2 A.$

3. $1 + \cot^2 A = \csc^2 A.$

NOTE. The equations in Examples 2 and 3 should be remembered.

4. $\cot A = \frac{\cos A}{\sin A}.$

7. $\cos A \csc A = \cot A.$

5. $\sin A \sec A = \tan A.$

8. $\tan A \cos A = \sin A.$

6. $\sin A \cos A = \frac{1}{2} \sin 2A.$

9. $\sin A \sec A \cot A = 1.$

10. $\cos A \csc A \tan A = 1.$
11. $(1 - \sin^2 A) \tan^2 A = \sin^2 A.$
12. $\sqrt{1 - \cos^2 A} \cot A = \cos A.$
13. $(1 + \tan^2 A) \sin^2 A = \tan^2 A.$
14. $(1 - \sin^2 A) \csc^2 A = \cot^2 A.$
15. $\tan^2 A \cos^2 A + \cos^2 A = 1.$
16. $(\sin^2 A - \cos^2 A)^2 = 1 - 4 \sin^2 A \cos^2 A.$
17. $(1 - \tan^2 A)^2 = \sec^4 A - 4 \tan^2 A.$
18. $\frac{\sin A}{\cos A} + \frac{\cos A}{\sin A} = \sec A \csc A.$
19. $\sin^4 A - \cos^4 A = \sin^2 A - \cos^2 A.$
20. $\sec A - \cos A = \sin A \tan A.$
21. $\csc A - \sin A = \cos A \cot A.$
22. $\frac{\cos A}{1 - \sin A} = \frac{1 + \sin A}{\cos A}.$

SECTION VII

APPLICATION OF FORMULAS [1], [2], [3]

Formulas [1], [2], and [3] enable us, when any one function of an angle is given, to find all the others. A given value of any one function, therefore, determines all the others.

EXAMPLE 1. Given $\sin A = \frac{3}{5}$; find the other functions.

By [1], p. 13, $\cos A = \sqrt{1 - \frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5} \sqrt{5}.$

By [2], p. 13, $\tan A = \frac{3}{4} \div \frac{4}{5} \sqrt{5} = \frac{3}{4} \times \frac{5}{4 \sqrt{5}} = \frac{15}{16 \sqrt{5}} = \frac{3}{4} \sqrt{5}.$

By [3], p. 14, $\cot A = \frac{4}{3} \sqrt{5}, \sec A = \frac{5}{4} \sqrt{5}, \csc A = \frac{5}{3}.$

EXAMPLE 2. Given $\tan A = 3$; find the other functions.

By [2], p. 13, $\frac{\sin A}{\cos A} = 3.$

And by [1], p. 13,

$$\sin^2 A + \cos^2 A = 1.$$

If we solve these equations (regarding $\sin A$ and $\cos A$ as two unknown quantities), we find

$$\sin A = \frac{3}{\sqrt{10}}, \cos A = \frac{1}{\sqrt{10}}.$$

Then, by [3], p. 14, $\cot A = \frac{1}{3}, \sec A = \sqrt{10}, \csc A = \sqrt{10}.$

EXAMPLE 3. Given $\sec A = m$; find the other functions.

By [3], p. 14, $\cos A = \frac{1}{m}.$

By [1], p. 13, $\sin A = \sqrt{1 - \frac{1}{m^2}} = \sqrt{\frac{m^2 - 1}{m^2}} = \frac{1}{m} \sqrt{m^2 - 1}.$

By [2], p. 13, $\tan A = \sqrt{m^2 - 1}.$

By [3], p. 14,

$$\cot A = \frac{1}{\sqrt{m^2 - 1}}, \csc A = \frac{m}{\sqrt{m^2 - 1}}.$$

EXERCISE VI

Find the values of the other functions, when :

1. $\sin A = \frac{1}{2}.$ 5. $\tan A = \frac{1}{3}.$ 9. $\csc A = \sqrt{2}.$

2. $\sin A = 0.8.$ 6. $\cot A = 1.$ 10. $\sin A = m.$

3. $\cos A = \frac{4}{5}.$ 7. $\cot A = 0.5.$ 11. $\sin A = \frac{2m}{1+m^2}.$

4. $\cos A = 0.28.$ 8. $\sec A = 2.$ 12. $\cos A = \frac{2mn}{m^2+n^2}.$

13. Given $\tan 45^\circ = 1$; find the other functions of 45° .

14. Given $\sin 30^\circ = \frac{1}{2}$; find the other functions of 30° .

15. Given $\csc 60^\circ = \frac{2}{\sqrt{3}}$; find the other functions of 60° .
 16. Given $\tan 15^\circ = 2 - \sqrt{3}$; find the other functions of 15° .
 17. Given $\cot 22^\circ 30' = \sqrt{2} + 1$; find the other functions of $22^\circ 30'$.

18. Given $\sin 0^\circ = 0$; find the other functions of 0° .
 19. Given $\sin 90^\circ = 1$; find the other functions of 90° .
 20. Given $\tan 90^\circ = \infty$; find the other functions of 90° .

Express the values of all the other functions in terms of:

21. $\sin A$. 22. $\cos A$. 23. $\tan A$. 24. $\cot A$.
 25. Given $2 \sin A = \cos A$; find $\sin A$ and $\cos A$.
 26. Given $4 \sin A = \tan A$; find $\sin A$ and $\tan A$.
 27. If $\sin A : \cos A = 9 : 40$, find $\sin A$ and $\cos A$.
 28. Transform the quantity $\tan^2 A + \cot^2 A - \sin^2 A - \cos^2 A$ into a form containing only $\cos A$.
 29. Prove that $\sin A + \cos A = (1 + \tan A) \cos A$.
 30. Prove that $\tan A + \cot A = \sec A \times \csc A$.

SECTION VIII

FUNCTIONS OF 45°

Let ACB (Fig. 8) be an isosceles right triangle, in which the length of the hypotenuse AB is equal to 1; then AC is equal to BC , and the angle A is equal to 45° . Since $\overline{AC}^2 + \overline{BC}^2 = 1$, therefore $2 \overline{AC}^2 = 1$, and $AC = \sqrt{\frac{1}{2}} = \frac{1}{2} \sqrt{2}$.

Therefore, by Sect. II, p. 5,

$$\sin 45^\circ = \cos 45^\circ = \frac{1}{2} \sqrt{2};$$

$$\tan 45^\circ = \cot 45^\circ = 1;$$

$$\sec 45^\circ = \csc 45^\circ = \sqrt{2}.$$

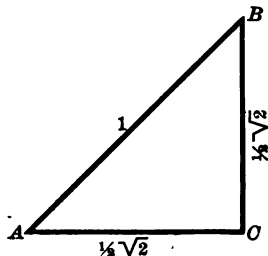


FIG. 8

SECTION IX

FUNCTIONS OF 30° AND 60°

Let ABC (Fig. 9) be an equilateral triangle, in which the length of each side is equal to 1; and let CD bisect the angle C . Then CD is perpendicular to AB and bisects AB .

Hence, $AD = \frac{1}{2}$, and $CD = \sqrt{1 - \frac{1}{4}} = \sqrt{\frac{3}{4}} = \frac{1}{2}\sqrt{3}$.

In the right triangle ADC , the angle $ACD = 30^\circ$, and the angle $CAD = 60^\circ$. Whence, by Sect. II, p. 5,

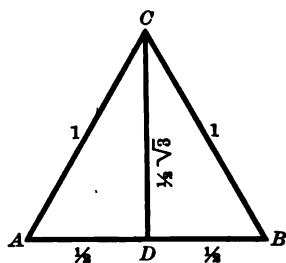


FIG. 9

$$\sin 30^\circ = \cos 60^\circ = \frac{1}{2};$$

$$\cos 30^\circ = \sin 60^\circ = \frac{1}{2}\sqrt{3};$$

$$\tan 30^\circ = \cot 60^\circ = \frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3};$$

$$\cot 30^\circ = \tan 60^\circ = \sqrt{3};$$

$$\sec 30^\circ = \csc 60^\circ = \frac{2}{\sqrt{3}} = \frac{2}{3}\sqrt{3};$$

$$\csc 30^\circ = \sec 60^\circ = 2.$$

The results for sine and cosine of 30° , 45° , and 60° may be easily remembered by arranging them in the following form:

Angle	30°	45°	60°	$\frac{1}{2}\sqrt{1} = 0.5$
Sine	$\frac{1}{2}\sqrt{1}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2} = 0.70711$
Cosine	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{1}$	$\frac{1}{2}\sqrt{3} = 0.86603$

EXERCISE VII

Solve the following equations:

1. $2 \cos x = \sec x.$

3. $\tan x = 2 \sin x.$

2. $4 \sin x = \csc x.$

4. $\sec x = \sqrt{2} \tan x.$

5. $\sin^2 x = 3 \cos^2 x$. 9. $\sin^2 x - \cos x = \frac{1}{4}$.
6. $2 \sin^2 x + \cos^2 x = \frac{3}{2}$. 10. $\tan^2 x - \sec x = 1$.
7. $3 \tan^2 x - \sec^2 x = 1$. 11. $\sin x + \sqrt{3} \cos x = 2$.
8. $\tan x + \cot x = 2$. 12. $\tan^2 x + \csc^2 x = 3$.
13. $2 \cos x + \sec x = 3$.
14. $\cos^2 x - \sin^2 x = \sin x$.
15. $2 \sin x + \cot x = 1 + 2 \cos x$.
16. $\sin^2 x + \tan^2 x = 3 \cos^2 x$.
17. $\tan x + 2 \cot x = \frac{5}{2} \csc x$.

NOTE. Wentworth & Hill's Five-place Logarithmic and Trigonometric Tables have full explanations, and directions for using them. Before proceeding to Chapter II the student should learn how to use these tables.

Table VI is to be used in solutions *without logarithms*. This four-place table contains the natural functions of angles at intervals of $1'$. The decimal point must be inserted before each value given, except when it appears in the values of the table.

CHAPTER II

THE RIGHT TRIANGLE

SECTION X

THE GIVEN PARTS

In order to solve a right triangle, two parts besides the right angle must be given, one of them at least being a side.

The two given parts may be :

- I. An acute angle and the hypotenuse.
- II. An acute angle and the opposite leg.
- III. An acute angle and the adjacent leg.
- IV. The hypotenuse and a leg.
- V. The two legs.

SECTION XI

SOLUTION WITHOUT LOGARITHMS

The following examples illustrate the process of solution when logarithms are not employed.

CASE I

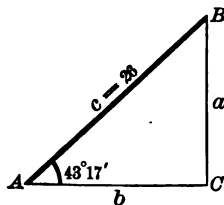


FIG. 10

Given $A = 43^\circ 17'$, $c = 26$; find B , a , b .

1. $B = 90^\circ - A = 46^\circ 43'$.
2. $\frac{a}{c} = \sin A$; $\therefore a = c \sin A$.
3. $\frac{b}{c} = \cos A$; $\therefore b = c \cos A$.

$$\begin{array}{r} \sin A = 0.6856 \\ c = \frac{26}{41136} \\ \frac{13712}{17.8256} \\ a = \end{array}$$

$$\begin{array}{r} \cos A = 0.7280 \\ c = \frac{26}{43680} \\ \frac{14560}{18.9280} \\ b = \end{array}$$

CASE II

Given $A = 13^\circ 58'$, $a = 15.2$; find B , b , c .

1. $B = 90^\circ - A = 76^\circ 2'$.

2. $\frac{b}{a} = \cot A$; $\therefore b = a \cot A$.

3. $\frac{a}{c} = \sin A$; $\therefore c = \frac{a}{\sin A}$.

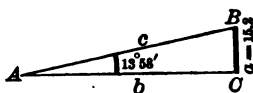


FIG. 11

$$\begin{array}{r} \cot A = 4.0207 \\ a = \frac{15.2}{80414} \\ \frac{201035}{40207} \\ b = 61.11464 \end{array}$$

$$\begin{array}{r} a = 15.2, \sin A = 0.2414. \\ 0.2414) 15.200 (62.9 \\ \underline{14\ 484} \\ 7160 \\ \underline{4828} \\ 2332 \\ c = 62.9 \end{array}$$

CASE III

Given $A = 27^\circ 12'$, $b = 31$; find B , a , c .

1. $B = 90^\circ - A = 62^\circ 48'$.

2. $\frac{a}{b} = \tan A$; $\therefore a = b \tan A$.

3. $\frac{b}{c} = \cos A$; $\therefore c = \frac{b}{\cos A}$.

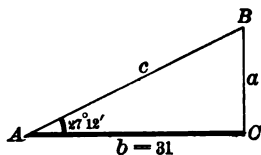


FIG. 12

$$\tan A = 0.5139$$

$$b = \frac{31}{0.5139}$$

$$a = 15.9309$$

$$b = 31, \cos A = 0.8894.$$

$$0.8894) 31.000 (34.9$$

$$\begin{array}{r} 26\ 682 \\ \hline 4\ 3180 \\ 3\ 5576 \\ \hline 7604 \end{array}$$

$$c = 34.9$$

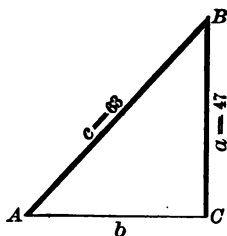


FIG. 13

$$a = 47, c = 63.$$

$$63) 47.0 (0.7460$$

$$\begin{array}{r} 44\ 1 \\ \hline 2\ 90 \\ 2\ 52 \end{array}$$

$$\sin A = 0.7460 \quad 380$$

$$\therefore A = 48^\circ 15' \quad 378$$

$$B = 41^\circ 45' \quad 2$$

$$c + a = 110$$

$$c - a = 16$$

$$660$$

$$110$$

$$b^2 = 1760$$

$$b = \sqrt{1760}$$

$$= 41.95$$

CASE IV

Given $a = 47, c = 63$; find A, B, b .

$$1. \sin A = \frac{a}{c}.$$

$$2. B = 90^\circ - A.$$

$$3. b = \sqrt{c^2 - a^2} = \sqrt{(c+a)(c-a)}.$$

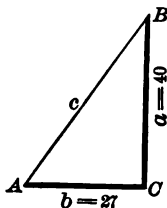


FIG. 14

CASE V

Given $a = 40, b = 27$; find A, B, c .

$$1. \tan A = \frac{a}{b}.$$

$$2. B = 90^\circ - A.$$

$$3. c = \sqrt{a^2 + b^2}.$$

$$a = 40, b = 27.$$

$$\frac{b}{a} = 1.4815$$

$$\tan A = 1.4815$$

$$\therefore A = 55^\circ 59'$$

$$B = 34^\circ 1'$$

$$a^2 = 1600$$

$$b^2 = 729$$

$$c^2 = 2329$$

$$\therefore c = \sqrt{2329}$$

$$= 48.26$$

SECTION XII

GENERAL METHOD OF SOLVING THE RIGHT TRIANGLE

From these five cases it appears that the general method of finding an unknown part in a right triangle is as follows:

Choose from the equation $A + B = 90^\circ$, and the equations that define the functions of the angles, an equation in which the required part only is unknown; solve this equation, if necessary, to find the value of the unknown part; then compute the value.

NOTE. In Case IV, if the given sides (here a and c) are nearly alike in value, then A is near 90° , and its value cannot be accurately found from the tables, because the sines of large angles differ little in value (as is evident from Fig. 4). In this case it is better to find B first, by means of the formula given on page 59, namely,

$$\tan \frac{1}{2} B = \sqrt{\frac{c-a}{c+a}}.$$

EXAMPLE. Given $a = 49$, $c = 50$; find A , B , b .

$$c - a = 1, c + a = 99.$$

$$\frac{c-a}{c+a} = 0.01010$$

$$\sqrt{\frac{c-a}{c+a}} = 0.1005$$

$$\tan \frac{1}{2} B = 0.1005$$

$$\therefore \frac{1}{2} B = 5^\circ 44'$$

$$B = 11^\circ 28'$$

$$A = 78^\circ 32'$$

$$c - a = 1$$

$$c + a = 99$$

$$c^2 - a^2 = 99$$

$$b^2 = 99$$

$$b = \sqrt{99}$$

$$= 9.95$$

EXERCISE VIII

1. In Case II give another way of finding c , after b has been found.
2. In Case III give another way of finding c , after a has been found.
3. In Case IV give another way of finding b , after the angles have been found.
4. In Case V give another way of finding c , after the angles have been found.
5. Given B and c ; find A , a , b .
6. Given B and b ; find A , a , c .
7. Given B and a ; find A , b , c .
8. Given b and c ; find A , B , a .

Solve the following right triangles:

	GIVEN	REQUIRED
9	$a = 3, \quad b = 4.$	$A = 36^\circ 52', \quad B = 53^\circ 8', \quad c = 5.$
10	$a = 7, \quad c = 13.$	$A = 32^\circ 35', \quad B = 57^\circ 25', \quad b = 10.954.$
11	$a = 5.3, \quad A = 12^\circ 17'.$	$B = 77^\circ 43', \quad b = 24.342, \quad c = 24.918.$
12	$a = 10.4, \quad B = 43^\circ 18'.$	$A = 46^\circ 42', \quad b = 9.800, \quad c = 14.290.$
13	$c = 26, \quad A = 37^\circ 42'.$	$B = 52^\circ 18', \quad a = 15.900, \quad b = 20.572.$
14	$c = 140, \quad B = 24^\circ 12'.$	$A = 65^\circ 48', \quad a = 127.694, \quad b = 57.386.$
15	$b = 19, \quad c = 23.$	$A = 34^\circ 18', \quad B = 55^\circ 42', \quad a = 12.961.$
16	$b = 98, \quad c = 135.2.$	$A = 43^\circ 33', \quad B = 46^\circ 27', \quad a = 93.139.$
17	$b = 42.4, \quad A = 32^\circ 14'.$	$B = 57^\circ 46', \quad a = 26.733, \quad c = 50.124.$
18	$b = 200, \quad B = 46^\circ 11'.$	$A = 43^\circ 49', \quad a = 191.900, \quad c = 277.160.$
19	$a = 95, \quad b = 37.$	$A = 68^\circ 43', \quad B = 21^\circ 17', \quad c = 101.951.$
20	$a = 6, \quad c = 103.$	$A = 3^\circ 21', \quad B = 86^\circ 39', \quad b = 102.825.$
21	$a = 3.12, \quad B = 5^\circ 8'.$	$A = 84^\circ 52', \quad b = 0.280, \quad c = 3.133.$
22	$a = 17, \quad c = 18.$	$A = 70^\circ 48', \quad B = 19^\circ 12', \quad b = 5.916.$
23	$c = 57, \quad A = 38^\circ 29'.$	$B = 51^\circ 31', \quad a = 35.471, \quad b = 44.620.$
24	$a + c = 18, \quad b = 12.$	$A = 22^\circ 37', \quad B = 67^\circ 23', \quad a = 5, \quad c = 13.$
25	$a + b = 9, \quad c = 8.$	$A = 82^\circ 18', \quad B = 7^\circ 42', \quad \begin{cases} a = 7.928, \\ b = 1.072. \end{cases}$

SECTION XIII

SOLUTION BY LOGARITHMS

CASE I

Given $A = 34^\circ 28'$, $c = 18.75$; find B , a , b .

1. $B = 90^\circ - A = 55^\circ 32'$.

2. $\frac{a}{c} = \sin A$; $\therefore a = c \sin A$.

3. $\frac{b}{c} = \cos A$; $\therefore b = c \cos A$.

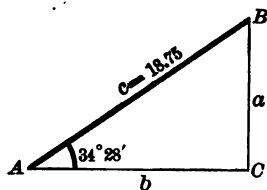


FIG. 15

$$\begin{array}{l} \log a = \log c + \log \sin A \\ \log c = 1.27300 \\ \log \sin A = 9.75276 - 10 \\ \hline \log a = 1.02576 \\ a = 10.611 \end{array}$$

$$\begin{array}{l} \log b = \log c + \log \cos A \\ \log c = 1.27300 \\ \log \cos A = 9.91617 - 10 \\ \hline \log b = 1.18917 \\ b = 15.459 \end{array}$$

CASE II

Given $A = 62^\circ 10'$, $a = 78$; find B , b , c .

1. $B = 90^\circ - A = 27^\circ 50'$.

2. $\frac{b}{a} = \cot A$; $\therefore b = a \cot A$.

3. $\frac{a}{c} = \sin A$.

$\therefore a = c \sin A$, and $c = \frac{a}{\sin A}$.

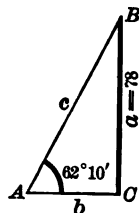


FIG. 16

$$\begin{array}{l} \log b = \log a + \log \cot A \\ \log a = 1.89209 \\ \log \cot A = 9.72262 - 10 \\ \hline \log b = 1.61471 \\ b = 41.182 \end{array}$$

$$\begin{array}{l} \log c = \log a + \operatorname{colog} \sin A \\ \log a = 1.89209 \\ \operatorname{colog} \sin A = 0.05340 \\ \hline \log c = 1.94549 \\ c = 88.204 \end{array}$$

CASE III

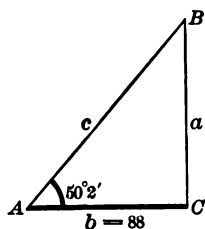


FIG. 17

Given $A = 50^\circ 2'$, $b = 88$; find B , a , c .

$$1. B = 90^\circ - A = 39^\circ 58'.$$

$$2. \frac{a}{b} = \tan A; \therefore a = b \tan A.$$

$$3. \frac{b}{c} = \cos A.$$

$$\therefore b = c \cos A, \text{ and } c = \frac{b}{\cos A}.$$

$$\begin{aligned} \log a &= \log b + \log \tan A \\ \log b &= 1.94448 \\ \log \tan A &= 10.07670 - 10 \\ \log a &= \frac{2.02118}{a = 105.00} \end{aligned}$$

$$\begin{aligned} \log c &= \log b + \text{colog} \cos A \\ \log b &= 1.94448 \\ \text{colog} \cos A &= 0.19223 \\ \log c &= \frac{2.13671}{c = 137.00} \end{aligned}$$

CASE IV

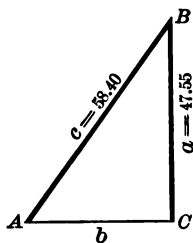


FIG. 18

Given $c = 58.40$, $a = 47.55$; find A , B , b .

$$1. \sin A = \frac{a}{c}.$$

$$2. B = 90^\circ - A.$$

$$3. \frac{b}{a} = \cot A; \therefore b = a \cot A.$$

$$\begin{aligned} \log \sin A &= \log a + \text{colog} c \\ \log a &= 1.67715 \\ \text{colog} c &= 8.23359 - 10 \\ \log \sin A &= \frac{9.91074 - 10}{A = 54^\circ 31'} \\ B &= 35^\circ 29' \end{aligned}$$

$$\begin{aligned} \log b &= \log a + \log \cot A \\ \log a &= 1.67715 \\ \log \cot A &= \frac{9.85300 - 10}{\log b = 1.53015} \\ b &= 33.896 \end{aligned}$$

CASE V

Given $a = 40$, $b = 27$; find A , B , c .

$$1. \tan A = \frac{a}{b}.$$

$$2. B = 90^\circ - A.$$

$$3. \frac{a}{c} = \sin A.$$

$$\therefore a = c \sin A, \text{ and } c = \frac{a}{\sin A}.$$

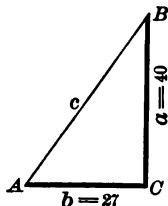


FIG. 19

$$\log \tan A = \log a + \text{colog } b$$

$$\log a = 1.60206$$

$$\text{colog } b = 8.56864 - 10$$

$$\log \tan A = 10.17070 - 10$$

$$A = 55^\circ 59'$$

$$B = 34^\circ 1'$$

$$\log c = \log a + \text{colog } \sin A$$

$$\log a = 1.60206$$

$$\text{colog } \sin A = 0.08151$$

$$\log c = 1.68357$$

$$c = 48.258$$

NOTE. In Cases IV and V the unknown side may also be found from the equations

$$(\text{for Case IV}) \quad b = \sqrt{c^2 - a^2} = \sqrt{(c + a)(c - a)};$$

$$(\text{for Case V}) \quad c = \sqrt{a^2 + b^2}.$$

These equations express the values of b and c directly in terms of the two given sides; and if the values of the sides are simple numbers (*e.g.*, 5, 12, 13), it is often easier to find b or c in this way. But this value of c is not adapted to logarithms, and this value of b is not so readily found by logarithms as the value of b given under Case IV. See also p. 23.

SECTION XIV

AREA OF THE RIGHT TRIANGLE

The area of a triangle is equal to one-half the product of the base by the altitude; therefore, if a and b denote the legs of a right triangle, and F the area, $F = \frac{1}{2}ab$.

Hence, the area may be found when a and b are known.

For example: Find the area, having given:

CASE I (Sect. XIII, p. 25).

$A = 34^\circ 28'$, $c = 18.75$.

First find (as in Sect. XIII, p. 25) $\log a$ and $\log b$.

$\log F = \log a + \log b + \text{colog } 2$

$\log a = 1.02576$

$\log b = 1.18917$

$\text{colog } 2 = 9.69897 - 10$

$\log F = 1.91390$

$F = 82.016$

CASE IV (Sect. XIII, p. 26).

$a = 47.55$, $c = 58.40$.

First find (as in Sect. XIII, p. 26) $\log a$ and $\log b$.

$\log F = \log a + \log b + \text{colog } 2$

$\log a = 1.67715$

$\log b = 1.53015$

$\text{colog } 2 = 9.69897 - 10$

$\log F = 2.90627$

$F = 805.88$

EXERCISE IX

Solve the following triangles by logarithms, finding the angles to the nearest minute:

	GIVEN		REQUIRED		
1	$a=6$,	$c=12$.	$A=30^\circ$,	$B=60^\circ$,	$b=10.392$.
2	$A=60^\circ$,	$b=4$.	$B=30^\circ$,	$c=8$,	$a=6.9282$.
3	$A=30^\circ$,	$a=3$.	$B=60^\circ$,	$c=6$,	$b=5.1961$.
4	$a=4$,	$b=4$.	$A=B=45^\circ$,	$c=5.6568$.	
5	$a=2$,	$c=2.82843$.	$A=B=45^\circ$,	$b=2$.	
6	$c=627$,	$A=23^\circ 30'$.	$B=66^\circ 30'$,	$a=250.02$,	$b=575.0$.
7	$c=2280$,	$A=28^\circ 5'$.	$B=61^\circ 55'$,	$a=1073.3$,	$b=2011.5$.
8	$c=72.15$,	$A=39^\circ 34'$.	$B=50^\circ 28'$,	$a=45.958$,	$b=55.620$.
9	$c=1$,	$A=36^\circ$.	$B=54^\circ$,	$a=0.58779$,	$b=0.80902$.
10	$c=200$,	$B=21^\circ 47'$.	$A=68^\circ 13'$,	$a=185.72$,	$b=74.22$.
11	$c=93.4$,	$B=76^\circ 25'$.	$A=13^\circ 35'$,	$a=21.936$,	$b=90.788$.
12	$a=637$,	$A=4^\circ 35'$.	$B=85^\circ 25'$,	$b=7946$,	$c=7971.5$.
13	$a=48.532$,	$A=36^\circ 44'$.	$B=53^\circ 16'$,	$b=65.031$,	$c=81.144$.
14	$a=0.0008$,	$A=86^\circ$.	$B=4^\circ$,	$b=0.0000559$,	$c=0.000802$.
15	$b=50.937$,	$B=43^\circ 48'$.	$A=46^\circ 12'$,	$a=53.116$,	$c=73.59$.
16	$b=2$,	$B=3^\circ 38'$.	$A=86^\circ 22'$,	$a=31.496$,	$c=31.559$.

	GIVEN		REQUIRED		
17	$a=992$,	$B=76^{\circ} 19'$.	$A=13^{\circ} 41'$,	$b=4074.5$,	$c=4193.5$.
18	$a=73$,	$B=68^{\circ} 52'$.	$A=21^{\circ} 8'$,	$b=188.86$,	$c=202.47$.
19	$a=2.189$,	$B=45^{\circ} 25'$.	$A=44^{\circ} 35'$,	$b=2.2211$,	$c=3.1185$.
20	$b=4$,	$A=37^{\circ} 56'$.	$B=52^{\circ} 4'$,	$a=3.1176$,	$c=5.0714$.
21	$c=8590$,	$a=4476$.	$A=31^{\circ} 24'$,	$B=58^{\circ} 30'$,	$b=7332.8$.
22	$c=86.53$,	$a=71.78$.	$A=56^{\circ} 3'$,	$B=33^{\circ} 57'$,	$b=48.324$.
23	$c=9.35$,	$a=8.49$.	$A=65^{\circ} 14'$,	$B=24^{\circ} 46'$,	$b=3.917$.
24	$c=2194$,	$b=1312.7$.	$A=53^{\circ} 15'$,	$B=36^{\circ} 45'$,	$a=1758$.
25	$c=30.69$,	$b=18.256$.	$A=53^{\circ} 30'$,	$B=36^{\circ} 30'$,	$a=24.67$.
26	$a=38.313$,	$b=19.522$.	$A=63^{\circ}$,	$B=27^{\circ}$,	$c=43$.
27	$a=1.2291$,	$b=14.950$.	$A=4^{\circ} 42'$,	$B=85^{\circ} 18'$,	$c=15$.
28	$a=415.38$,	$b=62.080$.	$A=81^{\circ} 30'$,	$B=8^{\circ} 30'$,	$c=420$.
29	$a=13.690$,	$b=16.926$.	$A=38^{\circ} 58'$,	$B=51^{\circ} 2'$,	$c=21.769$.
30	$c=91.92$,	$a=2.19$.	$A=1^{\circ} 22'$,	$B=88^{\circ} 38'$,	$b=91.894$.

Compute the unknown parts and also the area, having given :

31. $a = 5$, $b = 6$. 36. $c = 68$, $A = 69^{\circ} 54'$.
 32. $a = 0.615$, $c = 70$. 37. $c = 27$, $B = 44^{\circ} 4'$.
 33. $b = \sqrt[3]{2}$, $c = \sqrt{3}$. 38. $a = 47$, $B = 48^{\circ} 49'$.
 34. $a = 7$, $A = 18^{\circ} 14'$. 39. $b = 9$, $B = 34^{\circ} 44'$.
 35. $b = 12$, $A = 29^{\circ} 8'$. 40. $c = 8.462$, $B = 86^{\circ} 4'$.

41. Find the value of F in terms of c and A .
 42. Find the value of F in terms of a and A .
 43. Find the value of F in terms of b and A .
 44. Find the value of F in terms of a and c .
 45. Given $F = 58$, $a = 10$; solve the triangle.
 46. Given $F = 18$, $b = 5$; solve the triangle.
 47. Given $F = 12$, $A = 29^{\circ}$; solve the triangle.
 48. Given $F = 100$, $c = 22$; solve the triangle.

49. Find the angles of a right triangle if the hypotenuse is equal to three times one of the legs.

50. Find the legs of a right triangle if the hypotenuse is 6, and one angle is twice the other.

51. In a right triangle given c , and $A = nB$; find a and b .

52. In a right triangle the difference between the hypotenuse and the greater leg is equal to the difference between the two legs. Find the angles.

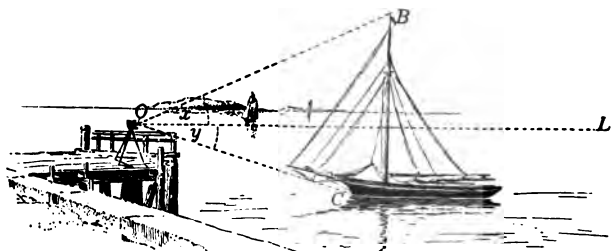


FIG. 20

The *angle of elevation* of an object, or the *angle of depression*, is the angle which a line from the eye to the object makes with a horizontal line in the same vertical plane.

Thus, if the observer is at O (Fig. 20), x is the angle of elevation of B , and y is the angle of depression of C .

53. At a horizontal distance of 120 feet from the foot of a steeple, the angle of elevation of the top was found to be $60^{\circ} 30'$. Find the height of the steeple.

54. From the top of a rock that rises vertically 326 feet out of the water, the angle of depression of a boat was found to be 24° . Find the distance of the boat from the foot of the rock.

55. How far is a monument, in a level plain, from the eye, if the height of the monument is 200 feet and the angle of elevation of the top $3^{\circ} 30'$?

56. A distance AB is measured 96 feet along the bank of a river from a point A opposite a tree C on the other bank. The angle ACB is $31^{\circ} 14'$. Find the breadth of the river.

57. What is the angle of elevation of an inclined plane if it rises 1 foot in a horizontal distance of 40 feet?

58. Find the angle of elevation of the sun when a tower 120 feet high casts a horizontal shadow 70 feet long.

59. How high is a tree that casts a horizontal shadow 80 feet in length when the angle of elevation of the sun is 50° ?

60. A ship is sailing due northeast at a rate of 10 miles an hour. Find the rate at which she is moving due north, and also due east.

61. In front of a window 20 feet high is a flower-bed 6 feet wide. How long is a ladder that will just reach from the edge of the bed to the window?

62. A ladder 40 feet long may be so placed that it will reach a window 33 feet high on one side of the street, and by turning it over without moving its foot it will reach a window 21 feet high on the other side. Find the breadth of the street.

63. From the top of a hill the angles of depression of two successive milestones, on a straight level road leading to the hill, are observed to be 5° and 15° . Find the height of the hill.

64. A fort stands on a horizontal plain. The angle of elevation at a certain point on the plain is 30° , and at a point 100 feet nearer the fort it is 45° . How high is the fort?

65. From a certain point on the ground the angles of elevation of the belfry of a church and of the top of the steeple were found to be 40° and 51° , respectively. From a point 300 feet farther off, on a horizontal line, the angle of elevation of the top of the steeple is found to be $33^\circ 45'$. Find the distance from the belfry to the top of the steeple.

66. The angle of elevation of the top C of an inaccessible fort observed from a point A is 12° . At a point B , 219 feet from A and on a line AB perpendicular to AC , the angle ABC is $61^\circ 45'$. Find the height of the fort:

SECTION XV

THE ISOSCELES TRIANGLE

An isosceles triangle is divided by the perpendicular from the vertex to the base into two *equal right triangles*.

Therefore, an isosceles triangle is determined by any two parts that determine one of these right triangles.

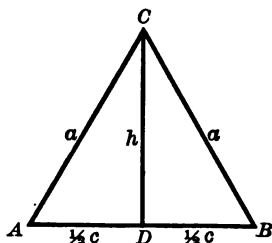


FIG. 21

Let the parts of an isosceles triangle CAB (Fig. 21), among which the altitude CD is to be included, be denoted as follows:

a = one of the equal sides,

c = the base,

h = the altitude,

A = one of the equal angles,

C = the angle at the vertex.

For example: Given a and c ; required A , C , h .

$$1. \cos A = \frac{\frac{1}{2}c}{a} = \frac{c}{2a}.$$

$$2. C + 2A = 180^\circ; \therefore C = 180^\circ - 2A = 2(90^\circ - A).$$

3. h may be found by any one of the equations:

$$h^2 + \frac{c^2}{4} = a^2;$$

whence

$$h = \sqrt{(a + \frac{1}{2}c)(a - \frac{1}{2}c)}.$$

$$\text{Also, } \frac{h}{a} = \sin A, \quad \text{and} \quad \frac{h}{\frac{1}{2}c} = \tan A;$$

whence

$$h = a \sin A, \quad \text{and} \quad h = \frac{1}{2}c \tan A.$$

When c and h are known, the area can be found by the formula

$$F = \frac{1}{2}ch.$$

27

EXERCISE X

Solve the following isosceles triangles, finding the angles to the nearest second :

1. Given a and A ; find C, c, h .
2. Given a and C ; find A, c, h .
3. Given c and A ; find C, a, h .
4. Given c and C ; find A, a, h .
5. Given h and A ; find C, a, c .
6. Given h and C ; find A, a, c .
7. Given a and h ; find A, C, c .
8. Given c and h ; find A, C, a .
9. Given $a = 14.3, c = 11$; find A, C, h .
10. Given $a = 0.295, A = 68^\circ 10'$; find c, h, F .
11. Given $c = 2.352, C = 69^\circ 49'$; find a, h, F .
12. Given $h = 7.4847, A = 76^\circ 14'$; find a, c, F .
13. Given $a = 6.71, h = 6.6$; find A, C, c .
14. Given $c = 9, h = 20$; find A, C, a .
15. Given $c = 147, F = 2572.5$; find A, C, a, h .
16. Given $h = 16.8, F = 43.68$; find A, C, a, c .
17. Find the value of F in terms of a and c .
18. Find the value of F in terms of a and C .
19. Find the value of F in terms of a and A .
20. Find the value of F in terms of h and C .
21. A barn is 40×80 feet, the pitch of the roof is 45° ; find the length of the rafters and the area of the whole roof.
22. In a unit circle what is the length of the chord corresponding to the angle 45° at the centre ?
23. If the radius of a circle is 30, and the length of a chord is 44, find the angle subtended at the centre.

24. Find the radius of a circle if a chord whose length is 5 subtends at the centre an angle of 133° .

25. What is the angle at the centre of a circle if the corresponding chord is equal to $\frac{3}{4}$ of the radius?

26. Find the area of a circular sector if the radius of the circle is 12, and the angle of the sector is 30° .

SECTION XVI

THE REGULAR POLYGON

Lines drawn from the centre of a regular polygon (Fig. 22) to the vertices are radii of the circumscribed circle; and lines drawn from the centre to the middle points of the sides are radii of the inscribed circle. These lines divide the polygon into *equal right triangles*. Therefore, a regular polygon is determined by a right triangle whose sides are the radius of the circumscribed circle, the radius of the inscribed circle, and half of one side of the polygon.

If the polygon has n sides, the angle of this right triangle at the centre of the polygon is equal to $\frac{1}{2}\left(\frac{360^\circ}{n}\right)$, or $\frac{180^\circ}{n}$; and the triangle may be solved when a side of the polygon or one of the radii is given.

Let

- n = number of sides,
- c = length of one side,
- r = radius of circumscribed circle,
- h = radius of inscribed circle,
- p = the perimeter,
- F = the area.

Then, by Geometry,

$$F = \frac{1}{2} hp.$$

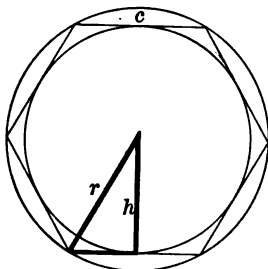


FIG. 22

EXERCISE XI

Find the remaining parts of a regular polygon, given :

1. $n = 10$, $c = 1$. 3. $n = 20$, $r = 20$. 5. $n = 11$, $F = 20$.
2. $n = 18$, $r = 1$. 4. $n = 8$, $h = 1$. 6. $n = 7$, $F = 7$.

Find the side of :

7. A regular decagon inscribed in a unit circle.
8. A regular decagon circumscribed about a unit circle.
9. If the side of an inscribed regular hexagon is 1, find the side of an inscribed regular dodecagon.
10. Given n and c , and let b denote the side of the inscribed regular polygon having $2n$ sides; find b in terms of n and c .
11. Compute the difference between the areas of a regular octagon and a regular nonagon if the perimeter of each is 16.
12. Compute the difference between the perimeters of a regular pentagon and a regular hexagon if the area of each is 12.

Find the area of :

13. The regular octagon formed by cutting away the corners of a square whose side is 1.
14. A regular pentagon if its diagonals are each equal to 12.
15. A regular polygon of 11 sides inscribed in a circle, if the area of an inscribed regular pentagon is 331.8.
16. A circle inscribed in an equilateral triangle whose perimeter is 20.
17. A regular polygon of 15 sides inscribed in a circle, if the area of a regular inscribed polygon of 16 sides is 100.
18. Find the perimeter of a regular dodecagon circumscribed about a circle the circumference of which is 1.
19. The area of a regular polygon of 25 sides is 40; find the area of the ring comprised between the circumferences of the inscribed and circumscribed circles.

CHAPTER III

GONIOMETRY

SECTION XVII

DEFINITION OF GONIOMETRY

To prepare the way for the solution of the oblique triangle, we now proceed to extend the definitions of the trigonometric functions to angles of all magnitudes, and to deduce certain useful relations of the functions of different angles.

That branch of Trigonometry which treats of trigonometric functions in general, and of their relations, is called *Goniometry*.

SECTION XVIII

POSITIVE AND NEGATIVE QUANTITIES

In measurements it is convenient to mark the distinction between two magnitudes that are measured in *opposite directions*, by calling one of them **positive** and the other **negative**.

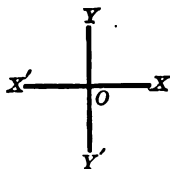


FIG. 23

Thus, if OX (Fig. 23) is considered to be positive, then OX' is considered to be negative; and if OY is considered to be positive, then OY' is considered to be negative.

When this distinction is applied to angles, an angle is considered to be *positive*, if the rotating line that describes it moves counter-clockwise, that is, in the direction opposite

to the hands of a clock, and to be *negative*, if the rotating line moves clockwise, that is, in the same direction as the hands of a clock.

Arcs corresponding to positive angles are considered *positive*, and arcs corresponding to negative angles are considered *negative*.

Thus, the angle AOB (Fig. 24) described by a line rotating about O from OA to OB is positive, and the arc AB is positive; the angle AOB' described by the line rotating about O from OA to OB' is negative, and the arc AB' is negative.

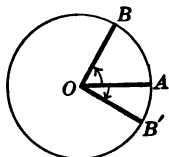


FIG. 24

SECTION XIX

CO-ORDINATES OF A POINT IN A PLANE

Let XX' (Fig. 25) be a horizontal line and let YY' be a line perpendicular to XX' at the point O . Then the plane determined by the lines XX' and YY' is divided into four quadrants which are numbered I, II, III, IV.

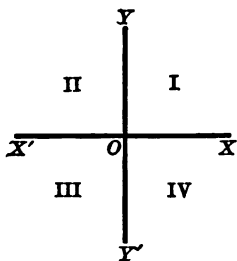


FIG. 25

Any point in the plane is determined by its *distance* and *direction* from each of the perpendiculars XX' and YY' . Its distance from YY' , measured on XX' , is called the **abscissa** of

the point; its distance from XX' , measured on YY' , is called the **ordinate** of the point.

The abscissa and the ordinate of a point are called the **co-ordinates** of the point; and the lines XX' and YY' are called the **axes of co-ordinates**. XX' is called the **axis of abscissas** or the **axis of x** ; YY' is called the **axis of ordinates** or the **axis of y** ; and the point O is called the **origin**.

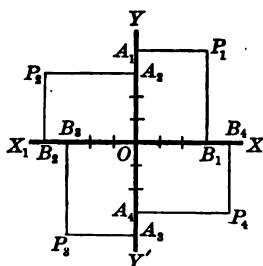


FIG. 26

In Fig. 26 the co-ordinates P_1 , P_2 , P_3 , P_4 are as follows :

The abscissa of P_1 is OB_1 ,
 and the ordinate of P_1 is OA_1 ;
 the abscissa of P_2 is OB_2 ,
 and the ordinate of P_2 is OA_2 ;
 the abscissa of P_3 is OB_3 ,
 and the ordinate of P_3 is OA_3 ;
 the abscissa of P_4 is OB_4 ,
 and the ordinate of P_4 is OA_4 .

Abscissas to the *right* of YY' are **positive**.

Abscissas to the *left* of YY' are **negative**.

Ordinates *above* XX' are **positive**.

Ordinates *below* XX' are **negative**.

Therefore,

in Quadrant I,

the abscissa is positive, the ordinate is positive;

in Quadrant II,

the abscissa is negative, the ordinate is positive;

in Quadrant III,

the abscissa is negative, the ordinate is negative;

in Quadrant IV,

the abscissa is positive, the ordinate is negative.

SECTION XX

ANGLES OF ANY MAGNITUDE

If the line OP (Figs. 27–30) is revolved about O from OX as its **initial position** counter-clockwise, as shown by the curved arrows, the line during one revolution will form with OX all angles from 0° to 360° .

Any particular angle is said to be an angle of that quadrant in which its **terminal side** lies.

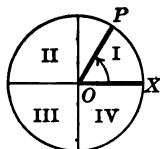


FIG. 27

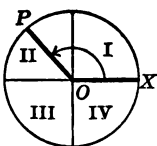


FIG. 28

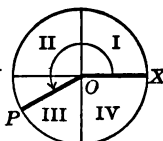


FIG. 29

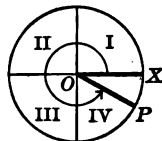


FIG. 30

Angles between 0° and 90° are angles of Quadrant I.
 Angles between 90° and 180° are angles of Quadrant II.
 Angles between 180° and 270° are angles of Quadrant III.
 Angles between 270° and 360° are angles of Quadrant IV.

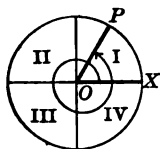


FIG. 31

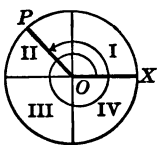


FIG. 32

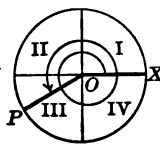


FIG. 33

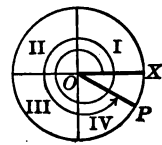


FIG. 34

If the revolving line makes another revolution (Figs. 31–34), it will describe all angles from 360° to 720° ; and so on.

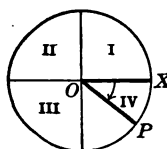


FIG. 35

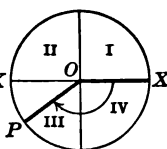


FIG. 36

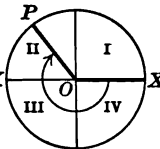


FIG. 37

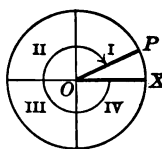


FIG. 38

If the line OP is revolved from OX clockwise (Figs. 35–38), it will describe all *negative* angles.

Thus we arrive at the conception of an angle of any magnitude, positive or negative.

SECTION XXI

FUNCTIONS OF ANY ANGLE

Figs. 39-42 show the functions in a unit circle drawn for an angle AOP in each quadrant, taken in order. The tangents to the circle are *always* drawn through A and B .

Let the angle AOP formed with OA by the moving radius OP be denoted by x ; then, in each quadrant,

$$\begin{array}{lll} \sin x = MP, & \tan x = AT, & \sec x = OT, \\ \cos x = OM, & \cot x = BS, & \csc x = OS. \end{array}$$

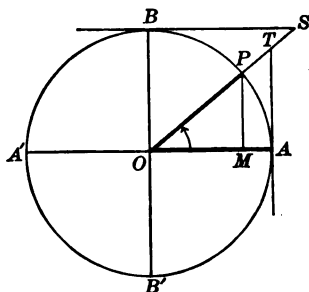


FIG. 39

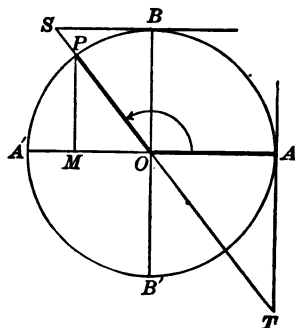


FIG. 40

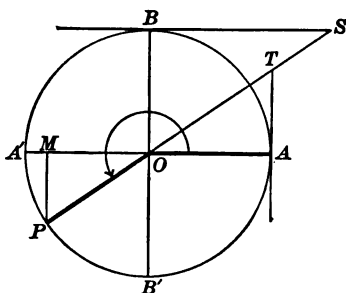


FIG. 41

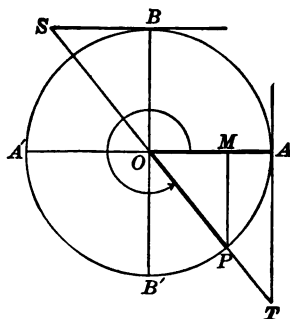


FIG. 42

If the terminal line of any angle x extends through the vertex indefinitely both ways, and if the circumference of a unit circle cuts the terminal line at P , the axis of abscissas at A , and the axis of ordinates at B , then

$\sin x$ = the ordinate of P ;

$\cos x$ = the abscissa of P ;

$\tan x$ = the tangent from A to meet the terminal line ;

$\cot x$ = the tangent from B to meet the terminal line ;

$\sec x$ = the segment of the terminal line between the vertex and the tangent ;

$\csc x$ = the segment of the terminal line between the vertex and the cotangent.

Sines and tangents extending from the axis of abscissas *upwards* are positive ; *downwards*, negative.

Cosines and cotangents extending from the axis of ordinates *towards the right* are positive ; *towards the left*, negative.

The signs of the secant and cosecant are determined by the signs of the cosine and sine, respectively. Therefore, secants and cosecants extending from the centre, *in the direction of the terminal line*, are considered positive ; *in the opposite direction*, negative. Hence,

QUADRANT	I	II	III	IV
$\sin$ and $\csc$	+	+	—	—
$\cos$ and $\sec$	+	—	—	+
$\tan$ and $\cot$	+	—	+	—

In Quadrant I *all* the functions are positive.

In Quadrant II the *sine and cosecant only* are positive.

In Quadrant III the *tangent and cotangent only* are positive.

In Quadrant IV the *cosine and secant only* are positive.

The signs of all the functions of any quadrant are known when the signs of the sine and cosine are known.

If the sine and cosine have like signs, the tangent and cotangent are positive; if unlike signs, negative. The sine and cosecant have like signs; the cosine and secant have like signs.

SECTION XXII

FUNCTIONS OF A VARIABLE ANGLE

Let the angle AOP (Fig. 43) increase continuously from 0° to 360° . The values of its functions change as follows:

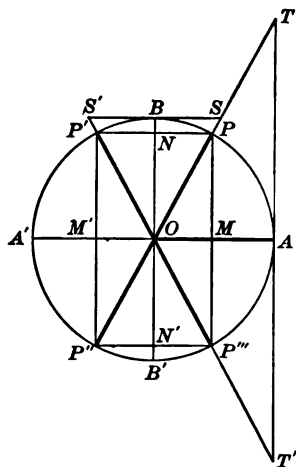


FIG. 43

1. *The Sine.* In the first quadrant the sine MP increases from 0 to 1; in the second it remains positive, and decreases from 1 to 0; in the third it is negative, and increases in absolute value from 0 to 1; in the fourth it is negative, and decreases in absolute value from 1 to 0.

2. *The Cosine.* In the first quadrant the cosine OM decreases from 1 to 0; in the second it becomes negative, and increases in absolute value from 0 to 1; in the third it is negative, and decreases in absolute value from 1

to 0; in the fourth it is positive, and increases from 0 to 1.

3. *The Tangent.* In the first quadrant the tangent AT increases from 0 to ∞ ; in the second it becomes negative, and decreases in absolute value from ∞ to 0; in the third it is positive, and increases from 0 to ∞ ; in the fourth it is negative, and decreases in absolute value from ∞ to 0.

4. *The Cotangent.* In the first quadrant, the cotangent BS decreases from ∞ to 0; in the second it is negative, and increases in absolute value from 0 to ∞ ; in the third and fourth quadrants, it has the same sign, and undergoes the same changes as in the first and second quadrants, respectively.

5. *The Secant.* In the first quadrant, the secant OT increases from 1 to ∞ ; in the second it is negative, and decreases in absolute value from ∞ to 1; in the third it is negative, and increases in absolute value from 1 to ∞ ; in the fourth it is positive, and decreases from ∞ to 1.

6. *The Cosecant.* In the first quadrant, the cosecant OS decreases from ∞ to 1; in the second it is positive, and increases from 1 to ∞ ; in the third it is negative, and decreases in absolute value from ∞ to 1; in the fourth it is negative, and increases in absolute value from 1 to ∞ .

The limiting values of the functions are as follows:

	0°	90°	180°	270°	360°
Sine	± 0	+1	± 0	-1	± 0
Cosine	+1	± 0	-1	± 0	+1
Tangent	± 0	$\pm \infty$	± 0	$\pm \infty$	± 0
Cotangent	$\pm \infty$	± 0	$\pm \infty$	± 0	$\pm \infty$
Secant	+1	$\pm \infty$	-1	$\pm \infty$	+1
Cosecant	$\pm \infty$	+1	$\pm \infty$	-1	$\pm \infty$

Sines and cosines vary in value from +1 to -1; tangents and cotangents, from $+\infty$ to $-\infty$; secants and cosecants, from $+\infty$ to +1, and from -1 to $-\infty$.

In the table given above the double sign $\pm$ is placed before 0 and ∞ . From the preceding investigation it appears that the functions *always change sign in passing through 0 and ∞* ; and the sign + or - prefixed to 0 or ∞ simply shows the direction from which the value is reached.

SECTION XXIII

FUNCTIONS OF ANGLES LARGER THAN 360°

The functions of $360^\circ + x$ are the same in sign and in absolute value as those of x ; for the moving radius has the same position in both cases. If n is a positive integer,

The functions of $(n \times 360^\circ + x)$ are the same as those of x .

For example: The functions of $2200^\circ (6 \times 360^\circ + 40^\circ)$ are equal to the functions of 40° .

SECTION XXIV

EXTENSION OF FORMULAS

The Formulas [1], [2], [3] established for *acute* angles on pp. 13, 14 hold true for *all* angles. Thus, in each quadrant

$$\overline{MP}^2 + \overline{OM}^2 = \overline{OP}^2.$$

Therefore,

$$\sin^2 x + \cos^2 x = 1. \quad [1]$$

We have in each quadrant from the similar triangles OMP , OAT , OBS the proportions

$$AT : OA = MP : OM,$$

$$\text{or } \tan x : 1 = \sin x : \cos x;$$

$$MP : OP = OB : OS,$$

$$\text{or } \sin x : 1 = 1 : \csc x;$$

$$OM : OP = OA : OT,$$

$$\text{or } \cos x : 1 = 1 : \sec x;$$

$$AT : OA = OB : BS,$$

$$\text{or } \tan x : 1 = 1 : \cot x.$$

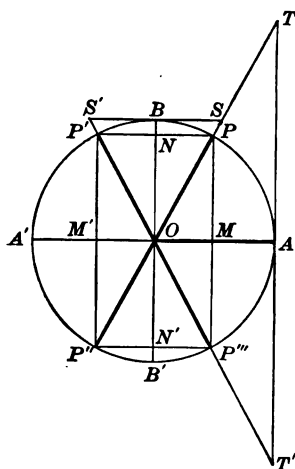


FIG. 44

$$\text{That is,} \quad \tan x = \frac{\sin x}{\cos x}. \quad [2]$$

$$\left. \begin{aligned} \sin x \times \csc x &= 1 \\ \cos x \times \sec x &= 1 \\ \tan x \times \cot x &= 1 \end{aligned} \right\}. \quad [3]$$

Formulas [1]–[3] enable us, from a given value of one function, to find the *absolute* values of the other five functions, and also the sign of the reciprocal function. But in order to determine the proper signs to be placed before the other four functions, we must know the quadrant to which the angle in question belongs, or the sign of any *one* of these four functions; for, by Sect. XXI, p. 40, it will be seen that *the signs of any two functions that are not reciprocals determine the quadrant to which the angle belongs.*

EXAMPLE. Given $\sin x = +\frac{4}{5}$, and $\tan x$ negative; find the values of the other functions.

Since $\sin x$ is positive, x is an angle in Quadrant I or II; but, since $\tan x$ is negative, Quadrant I is inadmissible.

$$\text{By [1],} \quad \cos x = \pm \sqrt{1 - \frac{16}{25}} = \pm \frac{3}{5}.$$

Since the angle is in Quadrant II, the minus sign must be taken, and we have

$$\cos x = -\frac{3}{5}.$$

By [2] and [3],

$$\tan x = -\frac{4}{3}, \quad \cot x = -\frac{3}{4}, \quad \sec x = -\frac{5}{3}, \quad \csc x = \frac{5}{4}.$$

EXERCISE XII

1. Construct the functions of an angle in Quadrant II. What are their signs?
2. Construct the functions of an angle in Quadrant III. What are their signs?

3. Construct the functions of an angle in Quadrant IV. What are their signs?

4. What are the signs of the functions of the following angles: 340° , 239° , 145° , 400° , 700° , 1200° , 3800° ?

5. How many angles less than 360° have the value of the sine equal to $+\frac{1}{2}$, and in what quadrants do they lie?

6. How many values less than 720° can the angle x have if $\cos x = +\frac{1}{2}$, and in what quadrants do they lie?

7. If we take into account only angles less than 180° , how many values can x have if $\sin x = \frac{1}{2}$? if $\cos x = \frac{1}{2}$? if $\cos x = -\frac{1}{2}$? if $\tan x = \frac{1}{2}$? if $\cot x = -7$?

8. Within what limits must the angle x lie if $\cos x = -\frac{1}{2}$? if $\cot x = 4$? if $\sec x = 80$? if $\csc x = -3$? (If $x < 360^\circ$.)

9. In what quadrant does an angle lie if sine and cosine are both negative? if cosine and tangent are both negative? if cotangent is positive and sine negative?

10. Between 0° and 3600° how many angles are there whose sines have the absolute value $\frac{1}{2}$? Of these sines how many are positive and how many negative?

11. In finding $\cos x$ by means of the equation $\cos x = \pm \sqrt{1 - \sin^2 x}$, when must we choose the positive sign and when the negative sign?

12. Given $\cos x = -\sqrt{\frac{1}{2}}$; find the other functions when x is an angle in Quadrant II.

13. Given $\tan x = \sqrt{3}$; find the other functions when x is an angle in Quadrant III.

14. Given $\sec x = +7$, and $\tan x$ negative; find the other functions of x .

15. Given $\cot x = -3$; find all the possible values of the other functions.

16. What functions of an angle of a triangle may be negative? In what case are they negative?

17. Why may $\cot 360^\circ$ be considered equal either to $+\infty$ or to $-\infty$?

18. Obtain by means of Formulas [1]–[3] the other functions of the angles, given:

$$(i) \tan 90^\circ = \infty. \quad (iii) \cot 270^\circ = 0.$$

$$(ii) \cos 180^\circ = -1. \quad (iv) \csc 360^\circ = -\infty.$$

19. Find the values of $\sin 450^\circ$, $\tan 540^\circ$, $\cos 630^\circ$, $\cot 720^\circ$, $\sin 810^\circ$, $\csc 900^\circ$.

Compute the values of the following expressions:

20. $a \sin 0^\circ + b \cos 90^\circ - c \tan 180^\circ$.

21. $a \cos 90^\circ - b \tan 180^\circ + c \cot 90^\circ$.

22. $a \sin 90^\circ - b \cos 360^\circ + (a - b) \cos 180^\circ$.

23. $(a^2 - b^2) \cos 360^\circ - 4 ab \sin 270^\circ$.

SECTION XXV

REDUCTION OF FUNCTIONS TO THE FIRST QUADRANT

In a unit circle (Fig. 45) draw two diameters PR and QS equally inclined to the horizontal diameter AA' , or so that the angles AOP , $A'OQ$, $A'OR$, and AOS shall be equal. From the points P , Q , R , S let fall perpendiculars to AA' ; the four right triangles thus formed, with a common vertex at O , are equal; because they have equal hypotenuses (radii of the circle) and equal acute angles at O . Therefore, the perpendiculars PM , QN , RN , SM are equal, and are the sines of the angles AOP , AOQ , AOR , AOS , respectively.

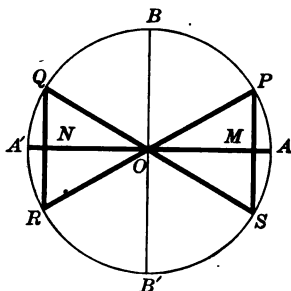


FIG. 45

Therefore, *in absolute value*,

$$\sin AOP = \sin AOQ = \sin AOR = \sin AOS.$$

And from Sect. XXIV, p. 44, it follows that *in absolute value* the cosines of these angles are also equal; and likewise the tangents, the cotangents, the secants, and the cosecants.*

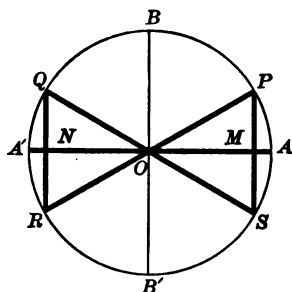


FIG. 46

Hence, *For every acute angle there is an angle in each of the higher quadrants whose functions, in absolute value, are equal to those of this acute angle.*

Let $\angle AOP = x$, $\angle POB = y$; then $x + y = 90^\circ$, and the functions of x are equal to the co-

named functions of y (Sect. V, p. 11); and

$$\angle AOQ \text{ (in Quadrant II)} = 180^\circ - x = 90^\circ + y,$$

$$\angle AOR \text{ (in Quadrant III)} = 180^\circ + x = 270^\circ - y,$$

$$\angle AOS \text{ (in Quadrant IV)} = 360^\circ - x = 270^\circ + y.$$

Hence, prefixing the proper sign (Sect. XXI, p. 40), we have:

ANGLE IN QUADRANT II

$$\sin (180^\circ - x) = \sin x. \quad \sin (90^\circ + y) = \cos y.$$

$$\cos (180^\circ - x) = -\cos x. \quad \cos (90^\circ + y) = -\sin y.$$

$$\tan (180^\circ - x) = -\tan x. \quad \tan (90^\circ + y) = -\cot y.$$

$$\cot (180^\circ - x) = -\cot x. \quad \cot (90^\circ + y) = -\tan y.$$

* In future, secants, cosecants, versed sines, and covered sines will be disregarded. Secants and cosecants may be found by Formula [3], versed sines and covered sines by VII and VIII, p. 5, if wanted, but they are seldom used in computations.

ANGLE IN QUADRANT III

$$\begin{array}{ll}
 \sin (180^{\circ} + x) = -\sin x. & \sin (270^{\circ} - y) = -\cos y. \\
 \cos (180^{\circ} + x) = -\cos x. & \cos (270^{\circ} - y) = -\sin y. \\
 \tan (180^{\circ} + x) = \tan x. & \tan (270^{\circ} - y) = \cot y. \\
 \cot (180^{\circ} + x) = \cot x. & \cot (270^{\circ} - y) = \tan y.
 \end{array}$$

ANGLE IN QUADRANT IV

$$\begin{array}{ll}
 \sin (360^{\circ} - x) = -\sin x. & \sin (270^{\circ} + y) = -\cos y. \\
 \cos (360^{\circ} - x) = \cos x. & \cos (270^{\circ} + y) = \sin y. \\
 \tan (360^{\circ} - x) = -\tan x. & \tan (270^{\circ} + y) = -\cot y. \\
 \cot (360^{\circ} - x) = -\cot x. & \cot (270^{\circ} + y) = -\tan y.
 \end{array}$$

REMARK. The tangents and cotangents may be found directly from the figure, or by Formula [2].

It is evident, from these formulas,

1. *The functions of all angles can be reduced to the functions of angles in the first quadrant, and therefore to functions of angles not greater than 45° (Sect. V, p. 11).*

2. *If an acute angle is added to or subtracted from 180° or 360° , the functions of the resulting angle are equal in absolute value to the like-named functions of the acute angle; but if an acute angle is added to or subtracted from 90° or 270° , the functions of the resulting angle are equal in absolute value to the co-named functions of the acute angle.*

3. *A given value of a sine or cosecant determines two supplementary angles, one acute, the other obtuse; a given value of any other function determines only one angle: acute if the value is positive, obtuse if the value is negative. [See functions of $(180^{\circ} - x)$.]*

SECTION XXVI

FUNCTIONS OF ANGLES THAT DIFFER BY 90°

The general form of two angles whose difference is 90° is x and $90^\circ + x$, and they must lie in adjoining quadrants. The relations between their functions were found in Sect. XXV, p. 48, but only for the case when x is acute. These relations, however, may be shown to hold true for all values of x .

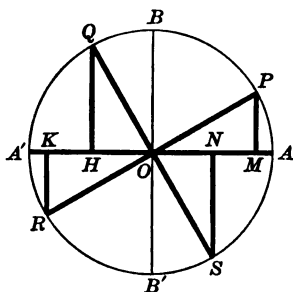


FIG. 47

In a unit circle (Fig. 47) draw two diameters PR and QS perpendicular to each other, and let fall to AA' the perpendiculars PM , QH , RK , and SN . The right triangles OMP , QHO , OKR , and SNO are equal, because they have equal hypotenuses and equal acute angles POM , OQH , ROK , and OSN .

Therefore, $OM = QH = OK = NS$,
and $PM = OH = RK = ON$.

Hence, taking into account the algebraic sign,

$$\begin{aligned}\sin AOQ &= \cos AOP; & \sin AOS &= \cos AOR; \\ \cos AOQ &= -\sin AOP; & \cos AOS &= -\sin AOR; \\ \sin AOR &= \cos AOQ; & \sin (360^\circ + AOP) &= \cos AOS; \\ \cos AOR &= -\sin AOQ; & \cos (360^\circ + AOP) &= -\sin AOS.\end{aligned}$$

In all these equations, if x denotes the angle on the right-hand side, the angle on the left-hand side is $90^\circ + x$.

Therefore, if x is an angle in any one of the four quadrants,

$$\begin{aligned}\sin (90^\circ + x) &= \cos x, & \tan (90^\circ + x) &= -\cot x. \\ \cos (90^\circ + x) &= -\sin x, & \cot (90^\circ + x) &= -\tan x.\end{aligned}$$

In like manner, it can be shown that all the formulas of Sect. XXV, p. 48, hold true, whatever the values of x and y .

Hence, *In every case the algebraic sign of the function of the resulting angle is the same as when x and y are both acute.*

SECTION XXVII

FUNCTIONS OF A NEGATIVE ANGLE

If the angle x is generated by the radius moving from the initial position OA to the terminal position OS , it will have the sign $-$, and its terminal side will be identical with its position for the angle $360^\circ - x$. Therefore, the functions of the angle $-x$ are the same as those of the angle $360^\circ - x$; or (Sect. XXV, p. 49),

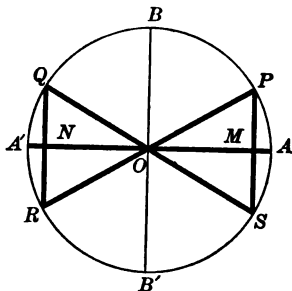


FIG. 48

$$\sin (-x) = -\sin x,$$

$$\cos (-x)=\cos x,$$

$$\tan (-x) = -\tan x,$$

$$\cot (-x) = -\cot x.$$

EXERCISE XIII

1. Express $\sin 250^\circ$ in terms of the functions of an acute angle less than 45° .

SOLUTION. $\sin 250^\circ = \sin (270^\circ - 20^\circ) = -\cos 20^\circ.$

Express the following functions in terms of the functions of angles less than 45° :

2. $\sin 172^\circ$.

5. $\cot 91^\circ$.

8. $\sin 204^\circ$.

3. $\cos 100^\circ$.

6. sec 110° .

9. $\cos 359^\circ$.

4. $\tan 125^\circ$.

7. $\csc 157^\circ$.

10. $\tan 300^\circ$.

- | | | |
|------------------------|----------------------------|----------------------------|
| 11. $\cot 264^\circ$. | 14. $\sin 163^\circ 49'$. | 17. $\cot 139^\circ 17'$. |
| 12. $\sec 244^\circ$. | 15. $\cos 195^\circ 33'$. | 18. $\sec 299^\circ 45'$. |
| 13. $\csc 271^\circ$. | 16. $\tan 269^\circ 15'$. | 19. $\csc 92^\circ 25'$. |

Express all the functions of the following negative angles in terms of the functions of positive angles less than 45° :

- | | | |
|--------------------|--------------------|------------------------|
| 20. -75° . | 22. -200° . | 24. $-52^\circ 37'$. |
| 21. -127° . | 23. -345° . | 25. $-196^\circ 54'$. |
26. Find the functions of 120° .

HINT. $120^\circ = 180^\circ - 60^\circ$, or $90^\circ + 30^\circ$; then apply Sect. XXV, p. 48.

Find the functions of the following angles:

- | | | | |
|-------------------|-------------------|-------------------|--------------------|
| 27. 135° . | 29. 210° . | 31. 240° . | 33. -30° . |
| 28. 150° . | 30. 225° . | 32. 300° . | 34. -225° . |

35. Given $\sin x = -\frac{1}{2}\sqrt{2}$, and $\cos x$ negative; find the other functions of x , and the value of x .

36. Given $\cot x = -\sqrt{3}$, and x in Quadrant II; find the other functions of x , and the value of x .

37. Find the functions of 3540° .

38. What angles less than 360° have a sine equal to $-\frac{1}{2}$? a tangent equal to $-\sqrt{3}$?

39. Which of the angles mentioned in Examples 27-34 have a cosine equal to $-\frac{1}{2}\sqrt{2}$? a cotangent equal to $-\sqrt{3}$?

40. What values of x between 0° and 720° will satisfy the equation $\sin x = +\frac{1}{2}$?

41. Find the other angle between 0° and 360° for which the corresponding function (sign included) has the same value as $\sin 12^\circ$, $\cos 26^\circ$, $\tan 45^\circ$, $\cot 72^\circ$, $\sin 191^\circ$, $\cos 120^\circ$, $\tan 244^\circ$, $\cot 357^\circ$.

42. Given $\tan 238^\circ = 1.6$; find $\sin 122^\circ$.
 43. Given $\cos 333^\circ = 0.89$; find $\tan 117^\circ$.

Simplify the following expressions:

44. $a \cos(90^\circ - x) + b \cos(90^\circ + x)$.
 45. $m \cos(90^\circ - x) \sin(90^\circ - x)$.
 46. $(a - b) \tan(90^\circ - x) + (a + b) \cot(90^\circ + x)$.
 47. $a^2 + b^2 - 2ab \cos(180^\circ - x)$.
 48. $\sin(90^\circ + x) \sin(180^\circ + x) + \cos(90^\circ + x) \cos(180^\circ - x)$.
 49. $\cos(180^\circ + x) \cos(270^\circ - y) - \sin(180^\circ + x) \sin(270^\circ - y)$.
 50. $\tan x + \tan(-y) - \tan(180^\circ - y)$.
 51. For what values of x is the expression $\sin x + \cos x$ positive, and for what values negative?
 52. Answer the questions of Example 51 for $\sin x - \cos x$.
 53. Find the functions of $x - 90^\circ$ in functions of x .
 54. Find the functions of $x - 180^\circ$ in functions of x .

SECTION XXVIII

FUNCTIONS OF THE SUM OF TWO ANGLES

In a unit circle (Fig. 49) let the angle $AOB = x$, the angle $BOC = y$; then the angle $AOC = x + y$.

In order to express $\sin(x + y)$ and $\cos(x + y)$ in terms of the sines and cosines of x and y , draw $CF \perp OA$, $CD \perp OB$, $DE \perp OA$, $DG \perp CF$; then $CD = \sin y$, $OD = \cos y$, and the angle $DCG = x$. Also,

$$\sin(x + y) = CF = DE + CG.$$

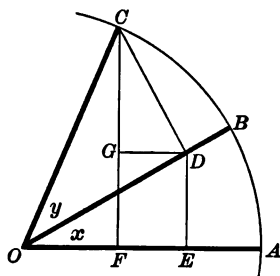


FIG. 49

Now $\frac{DE}{OD} = \sin x$; hence, $DE = \sin x \times OD = \sin x \cos y$.

And $\frac{CG}{CD} = \cos x$; hence,

$$CG = \cos x \times CD = \cos x \sin y.$$

Therefore,

$$\sin(x+y) = \sin x \cos y + \cos x \sin y. [4]$$

Again,

$$\cos(x+y) = OF = OE - DG.$$

$$\frac{OE}{OD} = \cos x; \text{ hence,}$$

$$OE = \cos x \times OD = \cos x \cos y.$$

$$\frac{DG}{CD} = \sin x; \text{ hence, } DG = \sin x \times CD = \sin x \sin y.$$

$$\text{Therefore, } \cos(x+y) = \cos x \cos y - \sin x \sin y. [5]$$

In this proof x and y , and also the sum $x+y$, are assumed to be acute angles. If the sum $x+y$ of the acute angles x and y is obtuse, as in Fig. 51, the proof remains, word for word, the same as above, the only difference being that the sign of OF will be negative, as DG is now greater than OE . The above formulas, therefore, hold true for all acute angles x and y .

If these formulas hold true for any two acute angles x and y , they hold true when one of the angles is increased by 90° . Thus, if for x we write $x' = 90^\circ + x$, then, by Sect. XXV, p. 48,

$$\sin(x' + y) = \sin(90^\circ + x + y) = \cos(x + y),$$

$$\cos(x' + y) = \cos(90^\circ + x + y) = -\sin(x + y).$$

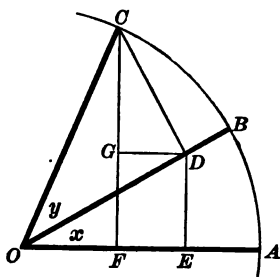


FIG. 50

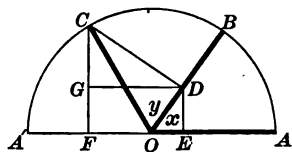


FIG. 51

Hence, by [5], $\sin(x' + y) = \cos x \cos y - \sin x \sin y$,
 by [4], $\cos(x' + y) = -\sin x \cos y - \cos x \sin y$.

Now, by Sect. XXV, p. 48,

$$\begin{aligned}\cos x &= \sin(90^\circ + x) = \sin x', \\ \sin x &= -\cos(90^\circ + x) = -\cos x' .\end{aligned}$$

Substitute these values of $\cos x$ and $\sin x$, then

$$\begin{aligned}\sin(x' + y) &= \sin x' \cos y + \cos x' \sin y, \\ \cos(x' + y) &= \cos x' \cos y - \sin x' \sin y.\end{aligned}$$

It follows that Formulas [4] and [5] hold true if either angle is repeatedly increased by 90° ; therefore they apply to all angles whatever.

By Sect. XXIV, p. 45, Formula [2],

$$\tan(x + y) = \frac{\sin(x + y)}{\cos(x + y)} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}.$$

If we divide each term of the numerator and denominator of the last fraction by $\cos x \cos y$, we have

$$\tan(x + y) = \frac{\frac{\sin x}{\cos x} + \frac{\sin y}{\cos y}}{1 - \frac{\sin x \sin y}{\cos x \cos y}}.$$

$$\text{That is, } \tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}. \quad [6]$$

$$\text{Also, } \cot(x + y) = \frac{\cos(x + y)}{\sin(x + y)} = \frac{\cos x \cos y - \sin x \sin y}{\sin x \cos y - \cos x \sin y}.$$

Divide each term of the numerator and denominator by $\sin x \sin y$, remembering that $\frac{\cos x}{\sin x} = \cot x$ and $\frac{\cos y}{\sin y} = \cot y$; we have

$$\cot(x + y) = \frac{\cot x \cot y - 1}{\cot y + \cot x}. \quad [7]$$

It is obvious that $(x - y) + y = x$. If we apply Formulas [4] and [5] to $(x - y) + y$, then

$$\sin \{(x - y) + y\} \text{ or } \sin x = \sin (x - y) \cos y + \cos (x - y) \sin y,$$

$$\cos \{(x - y) + y\} \text{ or } \cos x = \cos (x - y) \cos y - \sin (x - y) \sin y.$$

Multiply the first equation by $\cos y$, the second by $\sin y$,

$$\sin x \cos y = \sin (x - y) \cos^2 y + \cos (x - y) \sin y \cos y,$$

$$\cos x \sin y = -\sin (x - y) \sin^2 y + \cos (x - y) \sin y \cos y;$$

whence, by subtraction,

$$\sin x \cos y - \cos x \sin y = \sin (x - y) (\sin^2 y + \cos^2 y).$$

But $\sin^2 y + \cos^2 y = 1$ (Sect. XXIV, p. 44).

Therefore, by substitution and transposition,

$$\sin (x - y) = \sin x \cos y - \cos x \sin y.$$

Again, if we multiply the first equation by $\sin y$, the second equation by $\cos y$, and add the results, we obtain, by reducing,

$$\cos (x - y) = \cos x \cos y + \sin x \sin y.$$

Therefore, Formulas [8] and [9], like [4] and [5], from which they have been derived, are universally true.

From [8] and [9], by proceeding as in Sect. XXVIII, p. 55, we obtain

$$\tan (x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}. \quad [10]$$

$$\cot (x - y) = \frac{\cot x \cot y + 1}{\cot y - \cot x}. \quad [11]$$

Formulas [4]–[11] may be combined as follows:

$$\sin (x \pm y) = \sin x \cos y \pm \cos x \sin y,$$

$$\cos (x \pm y) = \cos x \cos y \mp \sin x \sin y,$$

$$\tan (x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y},$$

$$\cot (x \pm y) = \frac{\cot x \cot y \mp 1}{\cot y \pm \cot x}.$$

SECTION XXX

FUNCTIONS OF TWICE AN ANGLE

If $y = x$, Formulas [4]–[7] become

$$\sin 2x = 2 \sin x \cos x. \quad [12]$$

$$\cos 2x = \cos^2 x - \sin^2 x. \quad [13]$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}. \quad [14]$$

$$\cot 2x = \frac{\cot^2 x - 1}{2 \cot x}. \quad [15]$$

By these formulas the functions of twice an angle may be found when the functions of the angle are given.

SECTION XXXI

FUNCTIONS OF HALF AN ANGLE

$$\text{Formula [1] is } \cos^2 x + \sin^2 x = 1.$$

$$\text{Formula [13] is } \cos^2 x - \sin^2 x = \cos 2x.$$

$$\text{Subtract, } 2 \sin^2 x = 1 - \cos 2x.$$

$$\text{Add, } 2 \cos^2 x = 1 + \cos 2x.$$

Whence,

$$\sin x = \pm \sqrt{\frac{1 - \cos 2x}{2}}, \quad \cos x = \pm \sqrt{\frac{1 + \cos 2x}{2}}.$$

These values, if z is put for $2x$, and hence $\frac{1}{2}z$ for x , become

$$\sin \frac{1}{2}z = \pm \sqrt{\frac{1 - \cos z}{2}}. \quad [16]$$

$$\cos \frac{1}{2}z = \pm \sqrt{\frac{1 + \cos z}{2}}. \quad [17]$$

Hence, by division (Sect. XXIV, p. 45),

$$\tan \frac{1}{2} z = \pm \sqrt{\frac{1 - \cos z}{1 + \cos z}}. \quad [18]$$

$$\cot \frac{1}{2} z = \pm \sqrt{\frac{1 + \cos z}{1 - \cos z}}. \quad [19]$$

By these formulas the functions of half an angle may be computed when the cosine of the entire angle is given.

The proper sign to be placed before the root in each case depends on the quadrant in which the angle $\frac{1}{2} z$ lies (Sect. XXII, p. 42).

Let the student show from Formula [18] that

$$\tan \frac{1}{2} B = \sqrt{\frac{c - a}{c + a}}. \quad (\text{See p. 23, Note.})$$

SECTION XXXII

SUMS AND DIFFERENCES OF FUNCTIONS

From [4], [5], [8], and [9], by addition and subtraction,

$$\sin(x + y) + \sin(x - y) = 2 \sin x \cos y,$$

$$\sin(x + y) - \sin(x - y) = 2 \cos x \sin y,$$

$$\cos(x + y) + \cos(x - y) = 2 \cos x \cos y,$$

$$\cos(x + y) - \cos(x - y) = -2 \sin x \sin y;$$

or, by making $x + y = A$, and $x - y = B$, and, therefore, $x = \frac{1}{2}(A + B)$, and $y = \frac{1}{2}(A - B)$,

$$\sin A + \sin B = 2 \sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B). \quad [20]$$

$$\sin A - \sin B = 2 \cos \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B). \quad [21]$$

$$\cos A + \cos B = 2 \cos \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B). \quad [22]$$

$$\cos A - \cos B = -2 \sin \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B). \quad [23]$$

From [20] and [21], by division, we obtain

$$\frac{\sin A + \sin B}{\sin A - \sin B} = \tan \frac{1}{2}(A + B) \cot \frac{1}{2}(A - B);$$

or, since $\cot \frac{1}{2}(A - B) = \frac{1}{\tan \frac{1}{2}(A - B)},$

$$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan \frac{1}{2}(A + B)}{\tan \frac{1}{2}(A - B)}. \quad [24]$$

EXERCISE XIV

1. Find the value of $\sin(x + y)$ and $\cos(x + y)$ when $\sin x = \frac{3}{5}$, $\cos x = \frac{4}{5}$, $\sin y = \frac{1}{3}$, $\cos y = \frac{2}{3}$.

2. Find $\sin(90^\circ - y)$ and $\cos(90^\circ - y)$ by making $x = 90^\circ$ in Formulas [8] and [9].

Find, by Formulas [4]–[11], the first four functions of:

- | | | |
|---------------------|----------------------|---------------------|
| 3. $90^\circ + y.$ | 8. $360^\circ - y.$ | 13. $-y.$ |
| 4. $180^\circ - y.$ | 9. $360^\circ + y.$ | 14. $45^\circ - y.$ |
| 5. $180^\circ + y.$ | 10. $x - 90^\circ.$ | 15. $45^\circ + y.$ |
| 6. $270^\circ - y.$ | 11. $x - 180^\circ.$ | 16. $30^\circ + y.$ |
| 7. $270^\circ + y.$ | 12. $x - 270^\circ.$ | 17. $60^\circ - y.$ |

18. Find $\sin 3x$ in terms of $\sin x$.

19. Find $\cos 3x$ in terms of $\cos x$.

20. Given $\tan \frac{1}{2}x = 1$; find $\cos x$.

21. Given $\cot \frac{1}{2}x = \sqrt{3}$; find $\sin x$.

22. Given $\sin x = 0.2$; find $\sin \frac{1}{2}x$ and $\cos \frac{1}{2}x$.

23. Given $\cos x = 0.5$; find $\cos 2x$ and $\tan 2x$.

24. Given $\tan 45^\circ = 1$; find the functions of $22^\circ 30'$.

25. Given $\sin 30^\circ = 0.5$; find the functions of 15° .

26. Prove that $\tan 18^\circ = \frac{\sin 33^\circ + \sin 3^\circ}{\cos 33^\circ + \cos 3^\circ}.$

Prove the following formulas :

$$27. \sin 2x = \frac{2 \tan x}{1 + \tan^2 x} \qquad 29. \tan \frac{1}{2}x = \frac{\sin x}{1 + \cos x}.$$

$$28. \cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x} \qquad 30. \cot \frac{1}{2}x = \frac{\sin x}{1 - \cos x}.$$

$$31. \sin \frac{1}{2}x \pm \cos \frac{1}{2}x = \sqrt{1 \pm \sin x}.$$

$$32. \frac{\tan x \pm \tan y}{\cot x \pm \cot y} = \pm \tan x \tan y.$$

$$33. \tan (45^\circ - x) = \frac{1 - \tan x}{1 + \tan x}.$$

If A, B, C are the angles of a triangle, prove that :

$$34. \sin A + \sin B + \sin C = 4 \cos \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C.$$

$$35. \cos A + \cos B + \cos C = 1 + 4 \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C.$$

$$36. \tan A + \tan B + \tan C = \tan A \times \tan B \times \tan C.$$

$$37. \cot \frac{1}{2}A + \cot \frac{1}{2}B + \cot \frac{1}{2}C = \cot \frac{1}{2}A \times \cot \frac{1}{2}B \times \cot \frac{1}{2}C.$$

Change to a form more convenient for logarithmic computation :

$$38. \cot x + \tan x.$$

$$43. 1 + \tan x \tan y.$$

$$39. \cot x - \tan x.$$

$$44. 1 - \tan x \tan y.$$

$$40. \cot x + \tan y.$$

$$45. \cot x \cot y + 1.$$

$$41. \cot x - \tan y.$$

$$46. \cot x \cot y - 1.$$

$$42. \frac{1 - \cos 2x}{1 + \cos 2x}.$$

$$47. \frac{\tan x + \tan y}{\cot x + \cot y}.$$

SECTION XXXIII

ANTI-TRIGONOMETRIC FUNCTIONS

If y is any trigonometric function of an angle x , then x is said to be the corresponding *anti-trigonometric* function of y .

Thus, if $y = \sin x$, x is the *anti-sine* or *inverse sine* of y .

The anti-trigonometric functions of y are written

$$\begin{array}{cccc} \sin^{-1}y, & \tan^{-1}y, & \sec^{-1}y, & \text{vers}^{-1}y, \\ \cos^{-1}y, & \cot^{-1}y, & \csc^{-1}y, & \text{covers}^{-1}y. \end{array}$$

These are read, the angle whose sine is y , and so on.

For example, $\sin 30^\circ = \frac{1}{2}$; hence, $30^\circ = \sin^{-1}\frac{1}{2}$. Similarly, $90^\circ = \cos^{-1}0 = \sin^{-1}1$, and $45^\circ = \tan^{-1}1 = \sin^{-1}\frac{1}{\sqrt{2}}$, etc.

The symbol $^{-1}$ must not be confused with the exponent -1 . Thus, $\sin^{-1}x$ is a very different expression from $\frac{1}{\sin x}$, which would be written $(\sin x)^{-1}$. On the continent of Europe mathematical writers employ the notation *arc sin*, *arc cos*, etc., for $\sin^{-1}$, $\cos^{-1}$, etc.

There is an important difference between the trigonometric and the anti-trigonometric functions. When an angle is given, its functions are all completely determined; but when one of the functions is given, the angle may have any one of an indefinite number of values. Thus, if $\sin y = \frac{1}{2}$, y may be 30° , or 150° , or either of these increased or diminished by any integral multiple of 360° or 2π , but cannot take any other values. Accordingly, $\sin^{-1}\frac{1}{2} = 30^\circ \pm 2n\pi$, or $150^\circ \pm 2n\pi$, where n is any positive integer. Similarly, $\tan^{-1}1 = 45^\circ \pm 2n\pi$ or $225^\circ \pm 2n\pi$; i.e., $\tan^{-1}1 = 45^\circ \pm n\pi$.

Since one of the angles whose sine is x and one of the angles whose cosine is x together make 90° , and since similar relations hold for the tangent and cotangent, for the secant and cosecant, and for the versed sine and covered sine, we have

$$\begin{array}{ll} \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}, & \sec^{-1}x + \csc^{-1}x = \frac{\pi}{2}, \\ \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}, & \text{vers}^{-1}x + \text{covers}^{-1}x = \frac{\pi}{2}, \end{array}$$

where it must be understood that each equation is true only for a particular choice of the various possible values of the functions. Thus, if x is positive, and if the angles are always taken in the first quadrant, the equations are correct.

EXERCISE XV

1. Find all the values of the following functions:

$$\sin^{-1} \frac{1}{2} \sqrt{3}, \quad \tan^{-1} \frac{1}{3} \sqrt{3}, \quad \text{vers}^{-1} \frac{1}{2}, \quad \cos^{-1}(-\frac{1}{2} \sqrt{2}),$$

$$\csc^{-1} \sqrt{2}, \quad \tan^{-1} \infty, \quad \sec^{-1} 2, \quad \cos^{-1}(-\frac{1}{2} \sqrt{3}).$$

2. Prove that

$$\sin^{-1}(-x) = -\sin^{-1}x; \quad \cos^{-1}(-x) = \pi - \cos^{-1}x.$$

3. If $\sin^{-1}x + \sin^{-1}y = \pi$, prove that $x = y$.

4. If $y = \sin^{-1} \frac{1}{3}$, find $\tan y$.

5. Prove that $\cos(\sin^{-1}x) = \sqrt{1-x^2}$.

6. Prove that $\cos(2 \sin^{-1}x) = 1 - 2x^2$.

7. Prove that $\tan(\tan^{-1}x + \tan^{-1}y) = \frac{x+y}{1-xy}$.

8. If $x = \sqrt{\frac{1}{2}}$, find all the values of $\sin^{-1}x + \cos^{-1}x$.

9. Prove that $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \sin^{-1}x$.

10. Find the value of $\sin(\tan^{-1} \frac{4}{3})$.

11. Find the value of $\cot(2 \sin^{-1} \frac{3}{5})$.

12. Find the value of $\sin(\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3})$.

13. If $\sin^{-1}x = 2 \cos^{-1}x$, find x .

14. Prove that $\tan(2 \tan^{-1}x) = \frac{2x}{1-x^2}$.

15. Prove that $\sin(2 \tan^{-1}x) = \frac{2x}{1+x^2}$.

CHAPTER IV

THE OBLIQUE TRIANGLE

SECTION XXXIV

LAW OF SINES

Let A, B, C denote the angles of a triangle ABC (Figs. 53 and 54), and a, b, c , respectively, the lengths of the opposite sides.

Draw $CD \perp AB$, and meeting AB (Fig. 53) or AB produced (Fig. 54) at D . Let $CD = h$.

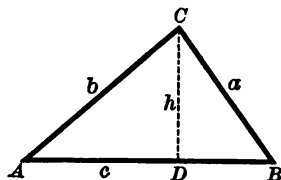


FIG. 53

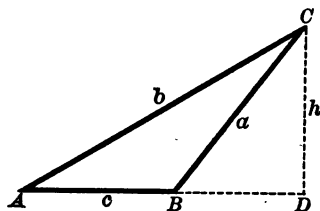


FIG. 54

In either figure, $\frac{h}{b} = \sin A$.

In Fig. 53, $\frac{h}{a} = \sin B$.

In Fig. 54, $\frac{h}{a} = \sin(180^\circ - B) = \sin B$.

Therefore, whether h lies within or without the triangle, we obtain, by division,

$$\frac{a}{b} = \frac{\sin A}{\sin B}. \quad [25]$$

By drawing perpendiculars from the vertices A and B to the opposite sides, we may obtain, in the same way,

$$\frac{b}{c} = \frac{\sin B}{\sin C}, \quad \frac{a}{c} = \frac{\sin A}{\sin C}.$$

Hence the Law of Sines:

The sides of a triangle are proportional to the sines of the opposite angles.

If we regard these three equations as proportions, and take them by alternation, it is evident that they may be written in the symmetrical form

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Each of these equal ratios has a simple geometrical meaning which will appear if the Law of Sines is proved as follows:

Circumscribe a circle about the triangle ABC (Fig. 55), and draw the radii OB , OC . Let R denote the radius. Draw $OM \perp BC$. By Geometry, the angle $BOC = 2A$; hence, the angle $BOM = A$, then

$$BM = R \sin BOM = R \sin A.$$

$$\therefore BC \text{ or } a = 2R \sin A.$$

In like manner,

$$b = 2R \sin B, \text{ and } c = 2R \sin C.$$

Whence we obtain

$$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

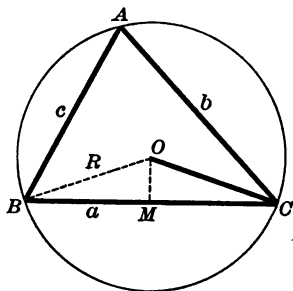


FIG. 55

That is: *The ratio of any side of a triangle to the sine of the opposite angle is numerically equal to the diameter of the circumscribed circle.*

SECTION XXXV

LAW OF COSINES

This law gives the value of one side of a triangle in terms of the other two sides and the angle included between them.

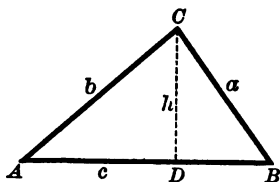


FIG. 56

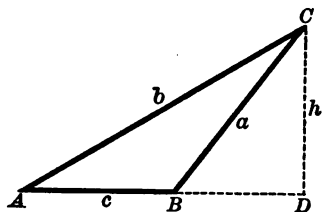


FIG. 57

In Figs. 56 and 57, $a^2 = h^2 + \overline{BD}^2$.

In Fig. 56, $BD = c - AD$.

In Fig. 57, $BD = AD - c$;

In either case, $\overline{BD}^2 = \overline{AD}^2 - 2c \times AD + c^2$.

Therefore, in all cases, $a^2 = h^2 + \overline{AD}^2 + c^2 - 2c \times AD$.

Now, $h^2 + \overline{AD}^2 = b^2$,

and $AD = b \cos A$.

Therefore, $a^2 = b^2 + c^2 - 2bc \cos A$. [26]

In like manner it may be proved that

$$b^2 = a^2 + c^2 - 2ac \cos B,$$

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

The three formulas have precisely the same form, and the Law of Cosines may be stated as follows:

The square of any side of a triangle is equal to the sum of the squares of the other two sides diminished by twice their product into the cosine of the included angle.

SECTION XXXVI

LAW OF TANGENTS

By Sect. XXXIV, p. 64, $a:b = \sin A : \sin B$;
whence, by the Theory of Proportion,

$$\frac{a-b}{a+b} = \frac{\sin A - \sin B}{\sin A + \sin B}.$$

But by [24], p. 60,

$$\frac{\sin A - \sin B}{\sin A + \sin B} = \frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)}.$$

Therefore,
$$\frac{a-b}{a+b} = \frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)}. \quad [27]$$

By merely changing the letters,

$$\frac{a-c}{a+c} = \frac{\tan \frac{1}{2}(A-C)}{\tan \frac{1}{2}(A+C)}, \quad \frac{b-c}{b+c} = \frac{\tan \frac{1}{2}(B-C)}{\tan \frac{1}{2}(B+C)}.$$

Hence the Law of Tangents:

The difference of two sides of a triangle is to their sum as the tangent of half the difference of the opposite angles is to the tangent of half their sum.

NOTE. If in [27] $b > a$, then $B > A$. The formula is still true, but to avoid negative numbers the formula in this case should be written

$$\frac{b-a}{b+a} = \frac{\tan \frac{1}{2}(B-A)}{\tan \frac{1}{2}(B+A)}.$$

EXERCISE XVI

1. What do the formulas of Sect. XXXIV, p. 64, become when one of the angles is a right angle?

2. Prove by means of the Law of Sines that the bisector of an angle of a triangle divides the opposite side into parts proportional to the adjacent sides.

3. What does Formula [26] become when $A = 90^\circ$? when $A = 0^\circ$? when $A = 180^\circ$? What does the triangle become in each of these cases?

NOTE. The case when $A = 90^\circ$ explains why the theorem of Sect. XXXV, p. 66, is sometimes called the *Generalized Theorem of Pythagoras*.

4. Prove (Figs. 56 and 57) that whether the angle B is acute or obtuse $c = a \cos B + b \cos A$. What are the two symmetrical formulas obtained by changing the letters? What does the formula become when $B = 90^\circ$?

5. From the three following equations (found in the last example) prove the theorem of Sect. XXXV, p. 66:

$$c = a \cos B + b \cos A,$$

$$b = a \cos C + c \cos A,$$

$$a = b \cos C + c \cos B.$$

HINT. Multiply the first equation by c , the second by b , the third by a ; then from the first subtract the sum of the second and third.

6. In Formula [27] what is the maximum value of $\frac{1}{2}(A-B)$?

7. Find the form to which Formula [27] reduces, and describe the nature of the triangle, when

(i) $C = 90^\circ$; (ii) $A - B = 90^\circ$, and $B = C$.

SECTION XXXVII

THE GIVEN PARTS

The formulas established in Sects. XXXIV-XXXVI, pp.

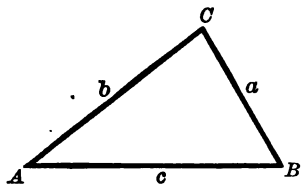


FIG. 58

64-67, together with the equation $A + B + C = 180^\circ$, are sufficient for solving every case of an oblique triangle. The three parts that determine an oblique triangle may be:

- I. One side and two angles;
- II. Two sides and the angle opposite one of these sides;
- III. Two sides and the included angle;
- IV. The three sides.

SECTION XXXVIII

SOLUTION OF AN OBLIQUE TRIANGLE

CASE I

Given one side a and two angles A and B ; find the remaining parts C , b , and c .

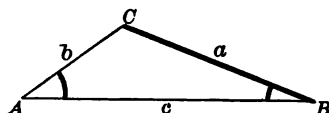


FIG. 59

$$1. C = 180^\circ - (A + B).$$

$$2. \frac{b}{a} = \frac{\sin B}{\sin A}; \quad \therefore b = \frac{a \sin B}{\sin A} = \frac{a}{\sin A} \times \sin B.$$

$$3. \frac{c}{a} = \frac{\sin C}{\sin A}; \quad \therefore c = \frac{a \sin C}{\sin A} = \frac{a}{\sin A} \times \sin C.$$

EXAMPLE. $a = 24.31$, $A = 45^\circ 18'$, $B = 22^\circ 11'$.

The work may be arranged as follows:

$a = 24.31$	$\log a = 1.38578$	$= 1.38578$
$A = 45^\circ 18'$	$\text{colog } \sin A = 0.14825$	$= 0.14825$
$B = 22^\circ 11'$	$\log \sin B = 9.57700$	$\log \sin C = 9.96556$
$A + B = 67^\circ 29'$	$\log b = 1.11103$	$\log c = 1.49959$
$C = 112^\circ 31'$	$b = 12.913$	$c = 31.593$

NOTE. When -10 is omitted after a logarithm or cologarithm, it must be remembered that the log or colog is 10 too large.

EXERCISE XVII

1. Given $a = 500$, $A = 10^\circ 12'$, $B = 46^\circ 36'$;
find $C = 123^\circ 12'$, $b = 2051.5$, $c = 2362.6$.
2. Given $a = 795$, $A = 79^\circ 59'$, $B = 44^\circ 41'$;
find $C = 55^\circ 20'$, $b = 567.69$, $c = 663.99$.

3. Given $a = 804$, $A = 99^\circ 55'$, $B = 45^\circ 1'$;
find $C = 35^\circ 4'$, $b = 577.31$, $c = 468.93$.
4. Given $a = 820$, $A = 12^\circ 49'$, $B = 141^\circ 59'$;
find $C = 25^\circ 12'$, $b = 2276.6$, $c = 1573.9$.
5. Given $c = 1005$, $A = 78^\circ 19'$, $B = 54^\circ 27'$;
find $C = 47^\circ 14'$, $a = 1340.6$, $b = 1113.8$.
6. Given $b = 13.57$, $B = 13^\circ 57'$, $C = 57^\circ 13'$;
find $A = 108^\circ 50'$, $a = 53.276$, $c = 47.324$.
7. Given $a = 6412$, $A = 70^\circ 55'$, $C = 52^\circ 9'$;
find $B = 56^\circ 56'$, $b = 5685.9$, $c = 5357.5$.
8. Given $b = 999$, $A = 37^\circ 58'$, $C = 65^\circ 2'$;
find $B = 77^\circ$, $a = 630.77$, $c = 929.48$.

9. In order to determine the distance of a hostile fort A from a place B , a line BC and the angles ABC and BCA were measured and found to be 322.55 yards, $60^\circ 34'$, and $56^\circ 10'$, respectively. Find the distance AB .

10. The angles B and C of a triangle ABC are $50^\circ 30'$ and $122^\circ 9'$, respectively, and BC is 9 miles. Find AB and AC .

11. Two observers 5 miles apart on a plain, and facing each other, find that the angles of elevation of a balloon in the same vertical plane with themselves are 55° and 58° , respectively. Find the distance from the balloon to each observer, and also the height of the balloon above the plain.

12. In a parallelogram given a diagonal d and the angles x and y which this diagonal makes with the sides; find the sides. Find the sides if $d = 11.237$, $x = 19^\circ 1'$, and $y = 42^\circ 54'$.

13. A lighthouse was observed from a ship to bear N. 34° E.; after the ship sailed due south 3 miles it bore N. 23° E. Find the distance from the lighthouse to the ship in each position.

NOTE. The phrase *to bear N. 34° E.* means that the line of sight to the lighthouse is in the northeast quarter of the horizon and makes, with a line due north, an angle of 34° .

14. In a trapezoid given the parallel sides a and b , and the angles x and y at the ends of one of the parallel sides; find the non-parallel sides. Compute the results when $a = 15$, $b = 7$, $x = 70^\circ$, $y = 40^\circ$.

Solve the following examples without using logarithms:

15. Given $b = 7.07107$, $A = 30^\circ$, $C = 105^\circ$; find a and c .

16. Given $c = 9.562$, $A = 45^\circ$, $B = 60^\circ$; find a and b .

17. The base of a triangle is 600 feet and the angles at the base are 30° and 120° . Find the other sides and the altitude.

18. Two angles of a triangle are, the one 20° , the other 40° . Find the ratio of the opposite sides.

19. The angles of a triangle are as $5 : 10 : 21$, and the side opposite the smallest angle is 3. Find the other sides.

20. Given one side of a triangle equal to 27, the adjacent angles equal each to 30° ; find the radius of the circumscribed circle. (See Sect. XXXIV, p. 65.)

SECTION XXXIX

CASE II

Given two sides a and b and the angle A opposite the side a ; find the remaining parts B , C , c .

This case, like the preceding case, is solved by means of the Law of Sines.

Since $\frac{\sin B}{\sin A} = \frac{b}{a}$, therefore $\sin B = \frac{b \sin A}{a}$;

$$C = 180^\circ - (A + B).$$

And since $\frac{c}{a} = \frac{\sin C}{\sin A}$, therefore $c = \frac{a \sin C}{\sin A}$.

When an angle is determined by its sine it admits of two values which are supplements of each other (Sect. XXV, p. 48); hence, either value of B may be taken unless excluded by the conditions of the problem.

If $a > b$, then by Geometry $A > B$, and B must be acute whatever be the value of A ; for a triangle can have only one obtuse angle. Hence, there is *one, and only one, triangle* that will satisfy the given conditions.

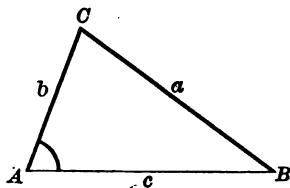


FIG. 60

If $a = b$, then by Geometry $A = B$; both A and B must be acute, and *the required triangle is isosceles*.

If $a < b$, then by Geometry $A < B$, and A must be acute in order that the triangle may be possible. If A is acute, it is evident from Fig. 61, where $\angle BAC = A$, $AC = b$, $CB = CB' = a$, that *the two triangles ACB and ACB' will satisfy the given conditions*, provided a is greater than the perpendicular CP ; that is, provided a is greater than $b \sin A$ (Sect. XI, p. 20). The angles ABC and $AB'C$ are supplementary (since $\angle ABC = \angle BB'C$); they are, in fact, the supplementary angles obtained from the formula

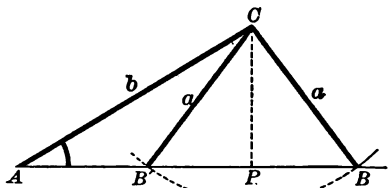


FIG. 61

$$\sin B = \frac{b \sin A}{a}.$$

If, however, $a = b \sin A = CP$ (Fig. 61), then $\sin B = 1$, $B = 90^\circ$, and *the triangle required is a right triangle*.

If $a < b \sin A$, that is, $< CP$, then $\sin B > 1$, and *the triangle is impossible*.

These results, for convenience, may be thus stated :

Two solutions ; if A is acute and the value of a lies between b and $b \sin A$.

No solution ; if A is acute and $a < b \sin A$;
or if A is obtuse and $a < b$, or $a = b$.

One solution ; in all other cases.

The number of solutions can often be determined by inspection. In case of doubt, find the value of $b \sin A$.

Or we may proceed to compute $\log \sin B$. If $\log \sin B = 0$, the triangle required is a right triangle. If $\log \sin B > 0$, the triangle is impossible. If $\log \sin B < 0$, there is *one solution* when $a > b$; there are *two solutions* when $a < b$.

When there are two solutions, let B' , C' , c' , denote the unknown parts of the second triangle ; then,

$$B' = 180^\circ - B, \quad C' = 180^\circ - (A + B') = B - A,$$

$$c' = \frac{a \sin C'}{\sin A}.$$

EXAMPLE 1. Given $a = 16$, $b = 20$, $A = 106^\circ$; find the remaining parts.

In this case $a < b$ and $A > 90^\circ$; therefore, the triangle is impossible.

EXAMPLE 2. Given $a = 36$, $b = 80$, $A = 30^\circ$; find the remaining parts.

Here we have $b \sin A = 80 \times \frac{1}{2} = 40$; so that $a < b \sin A$ and the triangle is impossible.

EXAMPLE 3. Given $a = 72,630$, $b = 117,480$, $A = 80^\circ 0' 50''$; find B , C , c .

$a = 72,630$	$\text{colog } a = 5.13888$	Here $\log \sin B > 0$. $\therefore$ no solution.
$b = 117,480$	$\log b = 5.06997$	
$A = 80^\circ 0' 50''$	$\log \sin A = 9.99337$	
	$\log \sin B = 0.20222$	

EXAMPLE 4. Given $a = 13.2$, $b = 15.7$, $A = 57^\circ 13' 15''$;
find B , C , c .

$a = 13.2$	$\text{colog } a = 8.87943$	$c = b \cos A$
$b = 15.7$	$\log b = 1.19590$	$\log b = 1.19590$
$A = 57^\circ 13' 15''$	$\log \sin A = 9.92467$	$\log \cos A = 9.73352$
Here $\log \sin B = 0$.	$\log \sin B = 0.00000$	$\log c = 0.92942$
$\therefore$ a <i>right triangle</i> .	$B = 90^\circ$	$c = 8.5$
	$\therefore C = 32^\circ 46' 45''$	

EXAMPLE 5. Given $a = 767$, $b = 242$, $A = 36^\circ 53' 2''$;
find B , C , c .

$a = 767$	$\text{colog } a = 7.11520$	$\log a = 2.88480$
$b = 242$	$\log b = 2.38382$	$\log \sin C = 9.86970$
$A = 36^\circ 53' 2''$	$\log \sin A = 9.77830$	$\text{colog } \sin A = 0.22170$
Here $a > b$,	$\log \sin B = 9.27732$	$\log c = 2.97620$
and $\log \sin B < 0$.	$B = 10^\circ 54' 58''$	$c = 946.68$
$\therefore$ one solution.	$\therefore C = 132^\circ 12' 0''$	

EXAMPLE 6. Given $a = 177.01$, $b = 216.45$, $A = 35^\circ 36' 20''$;
find the other parts.

$a = 177.01$	$\text{colog } a = 7.75200$	$\log a = 2.24800$	2.24800
$b = 216.45$	$\log b = 2.33536$	$\log \sin C = 9.99462$	9.23035
$A = 35^\circ 36' 20''$	$\log \sin A = 9.76507$	$\text{colog } \sin A = 0.23493$	0.23493
Here $a < b$,	$\log \sin B = 9.85243$	$\log c = 2.47755$	1.71328
and $\log \sin B < 0$.	$B = 45^\circ 23' 28''$	$c = 300.29$ or 51.675	
$\therefore$ two solutions.	or $134^\circ 36' 32''$		
	$\therefore C = 99^\circ 0' 12''$		
	or $9^\circ 47' 8''$		

EXERCISE XVIII

1. Find the number of solutions of the following:

- | | | |
|-------------------|---------------|----------------------|
| (i) $a = 80$, | $b = 100$, | $A = 30^\circ$. |
| (ii) $a = 50$, | $b = 100$, | $A = 30^\circ$. |
| (iii) $a = 40$, | $b = 100$, | $A = 30^\circ$. |
| (iv) $a = 13.4$, | $b = 11.46$, | $A = 77^\circ 20'$. |

- (v) $a = 70$, $b = 75$, $A = 60^\circ$.
 (vi) $a = 134.16$, $b = 84.54$, $B = 52^\circ 9' 11''$.
 (vii) $a = 200$, $b = 100$, $A = 30^\circ$.

2. Given $a = 840$, $b = 485$, $A = 21^\circ 31'$;
 find $B = 12^\circ 13' 34''$, $C = 146^\circ 15' 26''$, $c = 1272.1$.
3. Given $a = 9.399$, $b = 9.197$, $A = 120^\circ 35'$;
 find $B = 57^\circ 23' 40''$, $C = 2^\circ 1' 20''$, $c = 0.38525$.
4. Given $a = 91.06$, $b = 77.04$, $A = 51^\circ 9' 6''$;
 find $B = 41^\circ 13'$, $C = 87^\circ 37' 54''$, $c = 116.82$.
5. Given $a = 55.55$, $b = 66.66$, $B = 77^\circ 44' 40''$;
 find $A = 54^\circ 31' 13''$, $C = 47^\circ 44' 7''$, $c = 50.481$.
6. Given $a = 309$, $b = 360$, $A = 21^\circ 14' 25''$;
 find $B = 24^\circ 57' 54''$, $C = 133^\circ 47' 41''$, $c = 615.67$,
 $B' = 155^\circ 2' 6''$, $C' = 3^\circ 43' 29''$, $c' = 55.41$.
7. Given $a = 8.716$, $b = 9.787$, $A = 38^\circ 14' 12''$;
 find $B = 44^\circ 1' 28''$, $C = 97^\circ 44' 20''$, $c = 13.954$,
 $B' = 135^\circ 58' 32''$, $C' = 5^\circ 47' 16''$, $c' = 1.4202$.
8. Given $a = 4.4$, $b = 5.21$, $A = 57^\circ 37' 17''$;
 find $B = 90^\circ$, $C = 32^\circ 22' 43''$, $c = 2.7901$.
9. Given $a = 34$, $b = 22$, $B = 30^\circ 20'$;
 find $A = 51^\circ 18' 27''$, $C = 98^\circ 21' 33''$, $c = 43.098$,
 $A' = 128^\circ 41' 33''$, $C' = 20^\circ 58' 27''$, $c' = 15.593$.
10. Given $b = 19$, $c = 18$, $C = 15^\circ 49'$;
 find $B = 16^\circ 43' 13''$, $A = 147^\circ 27' 47''$, $a = 35.519$,
 $B' = 163^\circ 16' 47''$, $A' = 0^\circ 54' 13''$, $a' = 1.0415$.

11. Given $a = 75$, $b = 29$, $B = 16^\circ 15' 36''$; find the difference between the areas of the two corresponding triangles.

12. Given in a parallelogram the side a , a diagonal d , and the angle A made by the two diagonals; find the other diagonal.
 Special case: $a = 35$, $d = 63$, $A = 21^\circ 36' 30''$.

SECTION XL

CASE III

Given two sides a and b and the included angle C; find the remaining parts A, B, and c.

SOLUTION I. The angles A and B may both be found by means of Formula [27], Sect. XXXVI, p. 67, which may be written

$$\tan \frac{1}{2}(A - B) = \frac{a - b}{a + b} \times \tan \frac{1}{2}(A + B).$$

Since $\frac{1}{2}(A + B) = \frac{1}{2}(180^\circ - C)$, the value of $\frac{1}{2}(A + B)$ is known, so that this equation enables us to find the value of $\frac{1}{2}(A - B)$. We then have

$$\frac{1}{2}(A + B) + \frac{1}{2}(A - B) = A$$

and
$$\frac{1}{2}(A + B) - \frac{1}{2}(A - B) = B.$$

After A and B are known, the side c may be found by the Law of Sines, which gives its value in two ways, as follows:

$$c = \frac{a \sin C}{\sin A}, \quad \text{or} \quad c = \frac{b \sin C}{\sin B}.$$

SOLUTION II. The third side c may be found directly from the equation (Sect. XXXV, p. 66)

$$c = \sqrt{a^2 + b^2 - 2ab \cos C};$$

and then, by the Law of Sines, the following equations for computing the values of the angles A and B are obtained:

$$\sin A = a \times \frac{\sin C}{c}, \quad \sin B = b \times \frac{\sin C}{c}.$$

SOLUTION III. If, in the triangle ABC (Fig. 62), BD is drawn perpendicular to the side AC , then

$$\tan A = \frac{BD}{AD} = \frac{BD}{AC - DC}.$$

Now $BD = a \sin C$

and $DC = a \cos C.$

$$\therefore \tan A = \frac{a \sin C}{b - a \cos C}.$$

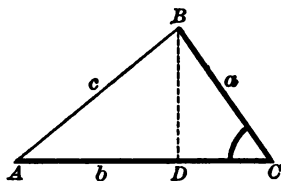


FIG. 62

By merely changing the letters,

$$\tan B = \frac{b \sin C}{a - b \cos C}.$$

It is not necessary, however, to use both formulas. When one angle, as A , has been found, the other, B , may be found from the relation $A + B + C = 180^\circ$.

When the angles are known, the third side is found by the Law of Sines, as in Solution I.

NOTE. When all three unknown parts are required, Solution I is the most convenient in practice. When only the third side, c , is desired, Solution II may be used to advantage, provided the values of a^2 and b^2 can be obtained readily without the aid of logarithms. But Solutions II and III are not adapted to logarithmic work.

EXAMPLE 1. Given $a = 748$, $b = 375$, $C = 63^\circ 35' 30''$; find A , B , and c .

$a + b = 1123$		
$a - b = 373$	$\log(a - b) = 2.57171$	$\log b = 2.57403$
$A + B = 116^\circ 24' 30''$	$\text{colog}(a + b) = 6.94962$	$\log \sin C = 9.95214$
$\frac{1}{2}(A + B) = 58^\circ 12' 15''$	$\log \tan \frac{1}{2}(A + B) = 0.20766$	$\text{colog} \sin B = 0.30073$
$\frac{1}{2}(A - B) = 28^\circ 10' 54''$	$\log \tan \frac{1}{2}(A - B) = 9.72899$	$\log c = 2.82690$
$A = 86^\circ 23' 9''$	$\frac{1}{2}(A - B) = 28^\circ 10' 54''$	$c = 671.27$
$B = 30^\circ 1' 21''$		

NOTE. In the above example we use the angle B in finding the side c rather than the angle A , because A is near 90° , and therefore the use of its sine should be avoided. See Note, p. 23.

EXAMPLE 2. Given $a=4$, $c=6$, $B=60^\circ$; find the third side b .

Here Solution II may be used to advantage. We have

$$b = \sqrt{a^2 + c^2 - 2ac \cos B} = \sqrt{16 + 36 - 24} = \sqrt{28};$$

$$\log 28 = 1.44716, \quad \log \sqrt{28} = 0.72358, \quad \sqrt{28} = 5.2915;$$

that is, $b = 5.2915$.

EXERCISE XIX

- Given $a = 77.99$, $b = 83.39$, $C = 72^\circ 15'$;
find $A = 51^\circ 15'$, $B = 56^\circ 30'$, $c = 95.24$.
- Given $b = 872.5$, $c = 632.7$, $A = 80^\circ$;
find $B = 60^\circ 45' 2''$, $C = 39^\circ 14' 58''$, $a = 984.83$.
- Given $a = 17$, $b = 12$, $C = 59^\circ 17'$;
find $A = 77^\circ 12' 53''$, $B = 43^\circ 30' 7''$, $c = 14.987$.
- Given $b = \sqrt{5}$, $c = \sqrt{3}$, $A = 35^\circ 53'$;
find $B = 93^\circ 28' 36''$, $C = 50^\circ 38' 24''$, $a = 1.3131$.
- Given $a = 0.917$, $b = 0.312$, $C = 33^\circ 7' 9''$;
find $A = 132^\circ 18' 27''$, $B = 14^\circ 34' 24''$, $c = 0.6775$.
- Given $a = 13.715$, $c = 11.214$, $B = 15^\circ 22' 36''$;
find $A = 118^\circ 55' 49''$, $C = 45^\circ 41' 35''$, $b = 4.1554$.
- Given $b = 3000.9$, $c = 1587.2$, $A = 86^\circ 4' 4''$;
find $B = 65^\circ 13' 51''$, $C = 28^\circ 42' 5''$, $a = 3297.2$.
- Given $a = 4527$, $b = 3465$, $C = 66^\circ 6' 27''$;
find $A = 68^\circ 29' 15''$, $B = 45^\circ 24' 18''$, $c = 4449$.
- Given $a = 55.14$, $b = 33.09$, $C = 30^\circ 24'$;
find $A = 117^\circ 24' 32''$, $B = 32^\circ 11' 28''$, $c = 31.431$.
- Given $a = 47.99$, $b = 33.14$, $C = 175^\circ 19' 10''$;
find $A = 4^\circ 46' 8''$, $B = 1^\circ 54' 42''$, $c = 81.066$.

11. If two sides of a triangle are each equal to 6, and the included angle is 60° , find the third side.

12. If two sides of a triangle are each equal to 6, and the included angle is 120° , find the third side.

13. Apply Solution I to the case in which a is equal to b ; that is, the case in which the triangle is isosceles.

14. If two sides of a triangle are 10 and 11, and the included angle is 50° , find the third side.

15. If two sides of a triangle are 43.301 and 25, and the included angle is 30° , find the third side.

16. In order to find the distance between two objects, A and B , separated by a swamp, a station C was chosen, and the distances $CA = 3825$ yards, $CB = 3475.6$ yards, together with the angle $ACB = 62^\circ 31'$, were measured. Find the distance from A to B .

17. Two inaccessible objects, A and B , are each viewed from two stations, C and D , on the same side of AB and 562 yards apart. The angle ACB is $62^\circ 12'$, BCD $41^\circ 8'$, ADB $60^\circ 49'$, and ADC $34^\circ 51'$; required the distance AB .

18. Two trains start at the same time from the same station and move along straight tracks that form an angle of 30° , one train at the rate of 30 miles an hour, the other at the rate of 40 miles an hour. How far apart are the trains at the end of half an hour?

19. In a parallelogram given the two diagonals 5 and 6, and the angle that they form $49^\circ 18'$; find the sides.

20. In a triangle one angle is $139^\circ 54'$, and the sides forming the angle have the ratio 5 : 9. Find the other two angles.

21. In order to find the distance between two objects, A and B , separated by a pond, a station C was chosen, and the distances $CA = 426$ yards, $CB = 322.4$ yards, together with the angle $ACB = 68^\circ 42'$, were measured. Find the distance from A to B .

SECTION XLI

CASE IV

Given the three sides a, b, c ; find the angles A, B, C .

The angles may be found directly from the formulas established in Sect. XXXV, p. 66. Thus, from the formula

$$a^2 = b^2 + c^2 - 2bc \cos A,$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

From this equation formulas adapted to logarithmic work are deduced as follows:

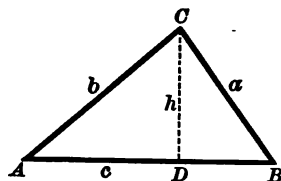


FIG. 63

For the sake of brevity, let

$$a + b + c = 2s;$$

then

$$b + c - a = 2(s - a),$$

$$a - b + c = 2(s - b),$$

and

$$a + b - c = 2(s - c).$$

Then the value of $1 - \cos A$ is

$$1 - \frac{b^2 + c^2 - a^2}{2bc} = \frac{2bc - b^2 - c^2 + a^2}{2bc} = \frac{a^2 - (b - c)^2}{2bc}$$

$$= \frac{(a + b - c)(a - b + c)}{2bc}$$

$$= \frac{2(s - b)(s - c)}{bc},$$

and the value of $1 + \cos A$ is

$$1 + \frac{b^2 + c^2 - a^2}{2bc} = \frac{2bc + b^2 + c^2 - a^2}{2bc} = \frac{(b + c)^2 - a^2}{2bc}$$

$$= \frac{(b + c + a)(b + c - a)}{2bc} = \frac{2s(s - a)}{bc}.$$

But from Formulas [16] and [17], p. 58, it follows that

$$1 - \cos A = 2 \sin^2 \frac{1}{2} A, \text{ and } 1 + \cos A = 2 \cos^2 \frac{1}{2} A.$$

$$\therefore 2 \sin^2 \frac{1}{2} A = \frac{2(s-b)(s-c)}{bc}, \text{ and } 2 \cos^2 \frac{1}{2} A = \frac{2s(s-a)}{bc},$$

$$\text{whence} \quad \sin \frac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{bc}}, \quad [28]$$

$$\cos \frac{1}{2} A = \sqrt{\frac{s(s-a)}{bc}}, \quad [29]$$

$$\text{and by [2],} \quad \tan \frac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}. \quad [30]$$

By merely changing the letters,

$$\sin \frac{1}{2} B = \sqrt{\frac{(s-a)(s-c)}{ac}}, \quad \sin \frac{1}{2} C = \sqrt{\frac{(s-a)(s-b)}{ab}}.$$

$$\cos \frac{1}{2} B = \sqrt{\frac{s(s-b)}{ac}}, \quad \cos \frac{1}{2} C = \sqrt{\frac{s(s-c)}{ab}}.$$

$$\tan \frac{1}{2} B = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}, \quad \tan \frac{1}{2} C = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}.$$

There is then a choice of three different formulas for finding the value of each angle. If half the angle is very near 0° , the formula for the cosine will not give a very accurate result, because the cosines of angles near 0° differ little in value; and the same holds true of the formula for the sine when half the angle is very near 90° . Hence, in the first case the formula for the sine, in the second that for the cosine, should be used.

But, in general, the formulas for the tangent are to be preferred.

It is not necessary to compute by the formulas more than two angles; for the third may then be found from the equation

$$A + B + C = 180^\circ.$$

There is this advantage, however, in computing all three angles by the formulas, that we may then use the sum of the angles as a test of the accuracy of the results.

In case it is desired to compute all the angles, the formulas for the tangent may be put in a more convenient form.

The value of $\tan \frac{1}{2} A$ may be written

$$\sqrt{\frac{(s-a)(s-b)(s-c)}{s(s-a)^2}}.$$

or
$$\frac{1}{s-a} \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

Hence, if we put

$$\sqrt{\frac{(s-a)(s-b)(s-c)}{s}} = r, \quad [31]$$

we have
$$\tan \frac{1}{2} A = \frac{r}{s-a}. \quad [32]$$

Likewise, $\tan \frac{1}{2} B = \frac{r}{s-b}, \quad \tan \frac{1}{2} C = \frac{r}{s-c}.$

EXAMPLE 1. Given $a = 3.41$, $b = 2.60$, $c = 1.58$; find the angles.

Using Formula [30] and the corresponding formula for $\tan \frac{1}{2} B$, we may arrange the work as follows:

$a = 3.41$	$\text{colog } s = 9.42079$	$\text{colog } s = 9.42079 - 10$
$b = 2.60$	$\text{colog } (s-a) = 0.41454$	$\log (s-a) = 9.58546 - 10$
$c = 1.58$	$\log (s-b) = 0.07737$	$\text{colog } (s-b) = 9.92263 - 10$
$2s = 7.59$	$\log (s-c) = 0.34537$	$\log (s-c) = 0.34537$
$s = 3.795$	$2 \overline{) 0.25807}$	$2 \overline{) 19.27425 - 20}$
$s-a = 0.385$	$\log \tan \frac{1}{2} A = 0.12903$	$\log \tan \frac{1}{2} B = 9.63713 - 10$
$s-b = 1.195$	$\frac{1}{2} A = 53^\circ 23' 20''$	$\frac{1}{2} B = 23^\circ 26' 37''$
$s-c = 2.215$	$A = 106^\circ 46' 40''$	$B = 46^\circ 53' 14''$

$$\therefore A + B = 153^\circ 39' 54'', \text{ and } C = 26^\circ 20' 6''.$$

EXAMPLE 2. Solve Example 1 by finding all three angles by the use of Formulas [31] and [32].

Here the work may be compactly arranged as follows, if we find $\log \tan \frac{1}{2}A$, etc., by *subtracting* $\log(s-a)$, etc., from $\log r$ instead of adding the cologarithm:

$a = 3.41$	$\log(s-a) = 9.58546$	$\log \tan \frac{1}{2}A = 10.12908$
$b = 2.60$	$\log(s-b) = 0.07737$	$\log \tan \frac{1}{2}B = 9.63713$
$c = 1.58$	$\log(s-c) = 0.34537$	$\log \tan \frac{1}{2}C = 9.36912$
$2s = 7.59$	$\text{colog } s = 9.42079$	$\frac{1}{2}A = 53^\circ 23' 20''$
$s = 3.795$	$\log r^2 = 9.42899$	$\frac{1}{2}B = 23^\circ 26' 37''$
$s-a = 0.385$	$\log r = 9.71450$	$\frac{1}{2}C = 13^\circ 10' 3''$
$s-b = 1.195$		$A = 106^\circ 46' 40''$
$s-c = 2.215$		$B = 46^\circ 53' 14''$
$2s = 7.590$ (check).		$C = 26^\circ 20' 6''$
		Check, $A + B + C = 180^\circ 0' 0''$

NOTE. Even if no mistakes are made in the work, the sum of the three angles found as above may differ very slightly from 180° in consequence of the fact that logarithmic computation is at best only a method of close approximation. When a difference of this kind exists, it should be divided among the angles according to the probable amount of error for each angle.

EXERCISE XX

Solve the following triangles, taking the three sides as the given parts:

	a	b	c	A	B	C
1	51	65	20	$38^\circ 52' 48''$	$126^\circ 52' 12''$	$14^\circ 15'$
2	78	101	29	$32^\circ 10' 55''$	$136^\circ 23' 50''$	$11^\circ 25' 15''$
3	111	145	40	$27^\circ 20' 32''$	$143^\circ 7' 48''$	$9^\circ 31' 40''$
4	21	26	31	$42^\circ 6' 13''$	$56^\circ 6' 36''$	$81^\circ 47' 11''$
5	19	34	49	$16^\circ 25' 36''$	$30^\circ 24'$	$133^\circ 10' 24''$
6	43	50	57	$46^\circ 49' 35''$	$57^\circ 59' 44''$	$75^\circ 10' 41''$
7	37	58	79	$26^\circ 0' 29''$	$43^\circ 25' 20''$	$110^\circ 34' 11''$
8	73	82	91	$49^\circ 34' 58''$	$58^\circ 46' 58''$	$71^\circ 38' 4''$
9	14.493	55.4363	66.9129	$8^\circ 20' 1''$	$33^\circ 40' 5''$	$137^\circ 59' 54''$
10	$\sqrt{5}$	$\sqrt{6}$	$\sqrt{7}$	$51^\circ 53' 12''$	$59^\circ 31' 48''$	$68^\circ 35'$

11. Given $a = 6$, $b = 8$, $c = 10$; find the angles.
12. Given $a = 6$, $b = 6$, $c = 10$; find the angles.
13. Given $a = 6$, $b = 6$, $c = 6$; find the angles.
14. Given $a = 6$, $b = 9$, $c = 12$; find the angles.
15. Given $a = 2$, $b = \sqrt{6}$, $c = \sqrt{3} - 1$; find the angles.
16. Given $a = 2$, $b = \sqrt{6}$, $c = \sqrt{3} + 1$; find the angles.
17. The distances between three cities, A , B , and C , are as follows: $AB = 165$ miles, $AC = 72$ miles, and $BC = 185$ miles. B is due east from A . In what direction is C from A ? What two answers are admissible?
18. Under what visual angle is an object 7 feet long seen by an observer whose eye is 5 feet from one end of the object and 8 feet from the other end?
19. When Formula [28] is used for finding the value of an angle, why does the ambiguity that occurs in Case II not exist?
20. If the sides of a triangle are 3, 4, and 6, find the sine of the largest angle.
21. Of three towns, A , B , and C , A is 200 miles from B and 184 miles from C , B is 150 miles due north from C . How far is A north of C ?
22. The sides of a triangle are 78.9, 65.4, 97.3, respectively. Find the largest angle.
23. The sides of a triangle are 487.25, 512.33, 544.37, respectively. Find the smallest angle.
24. Find the angles of a triangle whose sides are $\frac{\sqrt{3}+1}{2\sqrt{2}}$, $\frac{\sqrt{3}-1}{2\sqrt{2}}$, $\frac{\sqrt{3}}{2}$, respectively.
25. The sides of a triangle are 14.6 inches, 16.7 inches, and 18.8 inches, respectively. Find the length of the perpendicular from the vertex of the largest angle upon the opposite side.

SECTION XLII

AREA OF A TRIANGLE

CASE I

When two sides and the included angle are given,

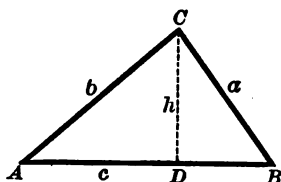


FIG. 64

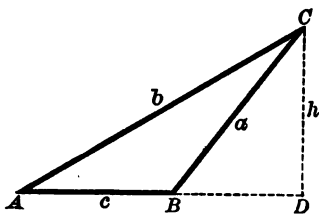


FIG. 65

In the triangle ABC (Fig. 64 or 65),

$$F = \frac{1}{2} c \times CD.$$

Now,

$$CD = a \sin B.$$

Therefore,

$$F = \frac{1}{2} ac \sin B.$$

[33]

Also, $F = \frac{1}{2} ab \sin C$, and $F = \frac{1}{2} bc \sin A$.

CASE II

When a side and the two adjacent angles are given.

$$\sin A : \sin C = a : c. \quad (\text{Sect. XXXIV, p. 65.})$$

Therefore,

$$c = \frac{a \sin C}{\sin A}.$$

Putting this value of c in Formula [33],

$$F = \frac{a^2 \sin B \sin C}{2 \sin A}.$$

But $\sin(B + C) = \sin(180^\circ - A) = \sin A$. (Sect. XXV, p. 48.)

Hence,

$$F = \frac{a^2 \sin B \sin C}{2 \sin(B + C)}. \quad [34]$$

CASE III

When the three sides of a triangle are given.

By Formula [12], p. 58,

$$\sin B = 2 \sin \frac{1}{2} B \times \cos \frac{1}{2} B.$$

Now, by Formula [28], p. 81,

$$\sin \frac{1}{2} B = \sqrt{\frac{(s-a)(s-c)}{ac}},$$

and by Formula [29], p. 81,

$$\cos \frac{1}{2} B = \sqrt{\frac{s(s-b)}{ac}}.$$

By substituting these values of $\sin \frac{1}{2} B$ and $\cos \frac{1}{2} B$ in the above equation, we have

$$\sin B = \frac{2}{ac} \sqrt{s(s-a)(s-b)(s-c)}.$$

By putting this value of $\sin B$ in [33], we have

$$F = \sqrt{s(s-a)(s-b)(s-c)}. \quad [35]$$

CASE IV

When the three sides and the radius of the circumscribed circle or the radius of the inscribed circle are given.

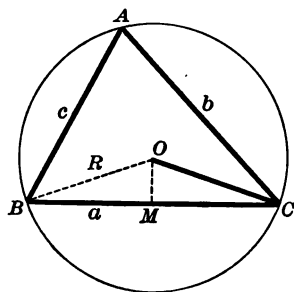


FIG. 66

If R denotes the radius of the circumscribed circle, we have, from Sect. XXXIV, p. 65,

$$\sin B = \frac{b}{2R}.$$

By putting this value of $\sin B$ in [33], we have

$$F = \frac{abc}{4R}. \quad [36]$$

If r denotes the radius of the inscribed circle, divide the triangle into three triangles by lines from the centre of this circle to the vertices; then the altitude of each of the three triangles is equal to r . Therefore,

$$F = \frac{1}{2} r (a + b + c) = rs. \quad [37]$$

By putting in this formula the value of F given in [35],

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}};$$

whence r , in [31], Sect. XLI, p. 82, is equal to the radius of the inscribed circle.

EXERCISE XXI

Find the area :

1. Given $a = 4474.5$, $b = 2164.5$, $C = 116^\circ 30' 20''$.
2. Given $b = 21.66$, $c = 36.94$, $A = 66^\circ 4' 19''$.
3. Given $a = 510$, $c = 173$, $B = 162^\circ 30' 28''$.
4. Given $a = 408$, $b = 41$, $c = 401$.
5. Given $a = 40$, $b = 13$, $c = 37$.
6. Given $a = 624$, $b = 205$, $c = 445$.
7. Given $b = 149$, $A = 70^\circ 42' 30''$, $B = 39^\circ 18' 28''$.
8. Given $a = 215.9$, $c = 307.7$, $A = 25^\circ 9' 31''$.
9. Given $b = 8$, $c = 5$, $A = 60^\circ$.
10. Given $a = 7$, $c = 3$, $A = 60^\circ$.
11. Given $a = 60$, $B = 40^\circ 35' 12''$, area = 12; find the radius of the inscribed circle.
12. Obtain a formula for the area of a parallelogram in terms of two adjacent sides and the included angle.
13. Obtain a formula for the area of an isosceles trapezoid in terms of the two parallel sides and an acute angle.

14. Two sides and included angle of a triangle are 2416, 1712, and 30° ; and two sides and included angle of another triangle are 1948, 2848, and 150° . Find the sum of their areas.

15. The base of an isosceles triangle is 20, and its area is $100 \div \sqrt{3}$; find its angles.

16. Show that the area of a quadrilateral is equal to one-half the product of its diagonals into the sine of their included angle.

EXERCISE XXII

1. From a ship sailing down the English Channel the Eddystone was observed to bear N. $33^\circ 45'$ W., and after the ship had sailed 18 miles S. $67^\circ 30'$ W. it bore N. $11^\circ 15'$ E. Find its distance from each position of the ship.

2. Two objects, *A* and *B*, were observed from a ship to be at the same instant in a line bearing N. 15° E. The ship then sailed northwest 5 miles, when it was found that *A* bore due east and *B* bore northeast. Find the distance from *A* to *B*.

3. A castle and a monument stand on the same horizontal plane. The angles of depression of the top and the bottom of the monument viewed from the top of the castle are 40° and 80° ; the height of the castle is 140 feet. Find the height of the monument.

4. If the sun's altitude is 60° , what angle must a stick make with the horizon in order that its shadow in a horizontal plane may be the longest possible?

5. If the sun's altitude is 30° , find the length of the longest shadow cast on a horizontal plane by a stick 10 feet in length.

6. In a circle with the radius 3 find the area of the part comprised between parallel chords whose lengths are 4 and 5. (Two solutions.)

CHAPTER V

MISCELLANEOUS EXAMPLES

PROBLEMS IN PLANE TRIGONOMETRY

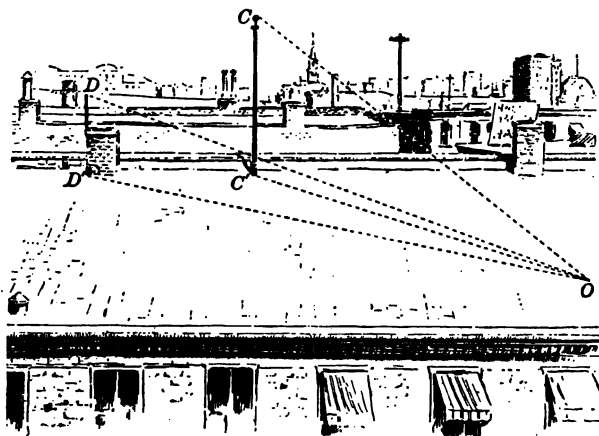


FIG. 67

If two objects are not in the same horizontal plane with each other or with the point of observation, we may suppose vertical lines to be passed through the two objects and to meet the horizontal plane of the point of observation in two points. The angular distance of these two points is the *bearing* of either of the objects from the other. Thus, the angle $C'OD'$ (Fig. 67) is the bearing of C from D .

NOTE. "Problems in Plane Trigonometry" are selected from those published by Mr. Charles W. Seaver, Cambridge, Mass. The full set can be obtained from him in pamphlet form.

EXERCISE XXIII

RIGHT TRIANGLES

1. The angle of elevation of a tower is $48^{\circ} 19' 14''$, and the distance of the base from the point of observation is 95 feet. Find the height of the tower and the distance of its top from the point of observation.

2. From a mountain 1000 feet high, the angle of depression of a ship is $77^{\circ} 35' 11''$. Find the distance of the ship from the summit of the mountain.

3. A flagstaff 90 feet high, on a horizontal plane, casts a shadow of 117 feet. Find the altitude of the sun.

4. When the moon is setting at any place, the angle at the moon subtended by the earth's radius passing through that place is $57' 3''$. If the earth's radius is 3956.2 miles, what is the moon's distance from the earth's centre?

5. The angle at the earth's centre subtended by the sun's radius is $16' 2''$ and the sun's distance is 92,400,000 miles. Find the sun's diameter in miles.

6. The latitude of Cambridge, Mass., is $42^{\circ} 22' 49''$. What is the length of the radius of that parallel of latitude?

7. At what latitude is the circumference of the parallel of latitude half of that of the equator?

8. In a circle with a radius of 6.7 is inscribed a regular polygon of thirteen sides. Find the length of one of its sides.

9. A regular heptagon, one side of which is 5.73, is inscribed in a circle. Find the radius of the circle.

10. A tower 93.97 feet high is situated on the bank of a river. The angle of depression of an object on the opposite bank is $25^{\circ} 12' 54''$. Find the breadth of the river.

11. From a tower 58 feet high the angles of depression of two objects situated in the same horizontal line with the base of the tower, and on the same side, are $30^{\circ} 13' 18''$ and $45^{\circ} 46' 14''$. Find the distance between these two objects.

12. Standing directly in front of one corner of a flat-roofed house, which is 150 feet in length, I observe that the horizontal angle which the length subtends has for its cosine $\sqrt{\frac{1}{2}}$, and that the vertical angle subtended by its height has for its sine $\frac{3}{\sqrt{34}}$. What is the height of the house?

13. A regular pyramid, with a square base, has a lateral edge 150 feet long, and a side of its base is 200 feet. Find the inclination of the face of the pyramid to the base.

14. From one edge of a ditch 36 feet wide, the angle of elevation of a wall on the opposite edge is $62^{\circ} 39' 10''$. Find the length of a ladder that will just reach from the point of observation to the top of the wall.

15. The top of a flagstaff has been partly broken off and touches the ground at a distance of 15 feet from the foot of the staff. If the length of the broken part is 39 feet, find the length of the whole staff.

16. From a balloon, which is directly above one town, is observed the angle of depression of another town, $10^{\circ} 14' 9''$. The towns being 8 miles apart, find the height of the balloon.

17. From the top of a mountain 3 miles high the angle of depression of the most distant object which is visible on the earth's surface is found to be $2^{\circ} 13' 50''$. Find the diameter of the earth.

18. A ladder 40 feet long reaches a window 33 feet high, on one side of a street. Being turned over upon its foot, it reaches another window 21 feet high, on the opposite side of the street. Find the width of the street.

19. The height of a house subtends a right angle at a window on the other side of the street; and the angle of elevation of the top of the house, from the same point, is 60° . The street is 30 feet wide. How high is the house?

20. A lighthouse 54 feet high is situated on a rock. The angle of elevation of the top of the lighthouse, as observed from a ship, is $4^\circ 52'$, and the angle of elevation of the top of the rock is $4^\circ 2'$. Find the height of the rock and its distance from the ship.

21. A man in a balloon observes the angle of depression of an object on the ground, bearing south, to be $35^\circ 30'$; the balloon drifts $2\frac{1}{2}$ miles east at the same height, when the angle of depression of the same object is $23^\circ 14'$. Find the height of the balloon.

22. A man standing south of a tower, on the same horizontal plane, observes its angle of elevation to be $54^\circ 16'$; he goes east 100 yards, and then finds its angle of elevation is $50^\circ 8'$. Find the height of the tower.

23. The angle of elevation of a tower at a place A, south of it, is 30° ; and at a place B, west of A, and at a distance of a from it, the angle of elevation is 18° . Show that the height of the tower is $\frac{a}{\sqrt{2} + 2\sqrt{5}}$; the tangent of 18° being $\frac{\sqrt{5} - 1}{\sqrt{10} + 2\sqrt{5}}$.

24. A pole is fixed on the top of a mound, and the angles of elevation of the top and the bottom of the pole are 60° and 30° , respectively. Prove that the length of the pole is twice the height of the mound.

25. At a distance a from the foot of a tower, the angle of elevation A of the top of the tower is the complement of the angle of elevation of a flagstaff on top of it. Show that the length of the staff is $2a \cot 2A$.

26. A *line of true level* is a line every point of which is equally distant from the centre of the earth. A line drawn tangent to a line of true level at any point is a line of *apparent level*. If at any point both these lines are drawn, and extended one mile, find the distance they are then apart.

27. In Problem 1, page 90, determine the effect upon the computed height of the tower of an error in either the angle of elevation or the measured distance.

OBLIQUE TRIANGLES

28. To determine the height of an inaccessible object situated on a horizontal plane, by observing its angles of elevation at two points in the same line with its base, and measuring the distance between these two points.

29. The angle of elevation of an inaccessible tower situated on a horizontal plane is $63^{\circ} 26'$; at a point 500 feet farther from the base of the tower the angle of elevation of its top is $32^{\circ} 14'$. Find the height of the tower.

30. A tower is situated on the bank of a river. From the opposite bank the angle of elevation of the tower is $60^{\circ} 13'$, and from a point 40 feet more distant the angle of elevation is $50^{\circ} 19'$. Find the breadth of the river.

31. A ship sailing north sees two lighthouses 8 miles apart, in a line due west; after an hour's sailing, one lighthouse bears S.W., and the other S.S.W. Find the ship's rate.

32. To determine the height of an accessible object situated on an inclined plane.

33. At the distance of 40 feet from the foot of a tower on an inclined plane, the tower subtends an angle of $41^{\circ} 19'$; at a point 60 feet farther away, the angle subtended by the tower is $23^{\circ} 45'$. Find the height of the tower.

34. A tower makes an angle of $113^{\circ} 12'$ with the inclined plane on which it stands ; and at a distance of 89 feet from its base, measured down the plane, the angle subtended by the tower is $23^{\circ} 27'$. Find the height of the tower.

35. From the top of a house 42 feet high the angle of elevation of the top of a pole is $14^{\circ} 13'$; at the bottom of the house it is $23^{\circ} 19'$. Find the height of the pole.

36. The sides of a triangle are 17, 21, 28. Prove that the length of a line bisecting the greatest side and drawn from the opposite angle is 13.

37. A privateer, 10 miles S.W. of a harbor, sees a ship sail from it in a direction S. 80° E., at a rate of 9 miles an hour. In what direction, and at what rate, must the privateer sail in order to come up with the ship in $1\frac{1}{2}$ hours ?

38. A person goes 70 yards up a slope of 1 in $3\frac{1}{2}$ from the edge of a river and observes the angle of depression of an object on the opposite bank to be $2\frac{1}{4}^{\circ}$. Find the breadth of the river.

39. The length of a lake subtends, at a certain point, an angle of $46^{\circ} 24'$, and the distances from this point to the two extremities of the lake are 346 and 290 feet. Find the length of the lake.

40. Two ships are a mile apart. The angular distance of the first ship from a fort on shore, as observed from the second ship, is $35^{\circ} 14' 10''$; the angular distance of the second ship from the fort, observed from the first ship, is $42^{\circ} 11' 53''$. Find the distance in feet from each ship to the fort.

41. Along the bank of a river is drawn a base line of 500 feet. The angular distance of one end of this line from an object on the opposite side of the river, as observed from the other end of the line, is 53° ; that of the second extremity

from the same object, observed at the first, is $79^{\circ} 12'$. Find the breadth of the river.

42. A vertical tower stands on a declivity inclined 15° to the horizon. A man ascends the declivity 80 feet from the base of the tower, and finds the angle then subtended by the tower to be 30° . Find the height of the tower.

43. The angle subtended by a tower on an inclined plane is; at a certain point, $42^{\circ} 17'$; 325 feet farther down it is $21^{\circ} 47'$. The inclination of the plane is $8^{\circ} 53'$. Find the height of the tower.

44. A cape bears north by east, as seen from a ship. The ship sails northwest 30 miles, and then the cape bears east. How far is it from the second point of observation?

45. Two observers, stationed on *opposite* sides of a cloud, observe its angles of elevation to be $44^{\circ} 56'$ and $36^{\circ} 4'$. Their distance from each other is 700 feet. What is the height of the cloud?

46. From a point B at the foot of a mountain, the angle of elevation of the top A is 60° . After ascending the mountain one mile, at an inclination of 30° to the horizon, and reaching a point C , the angle ACB is found to be 135° . Find the height of the mountain in feet.

47. From a ship two rocks are seen in the same right line with the ship, bearing N. 15° E. After the ship has sailed northwest 5 miles, the first rock bears east, and the second northeast. Find the distance between the rocks.

48. From a window on a level with the bottom of a steeple the angle of elevation of the steeple is 40° , and from a second window 18 feet higher the angle of elevation is $37^{\circ} 30'$. Find the height of the steeple.

49. To determine the distance between two inaccessible objects by observing angles at the extremities of a line of known length.

50. Wishing to determine the distance between a church A and a tower B , on the opposite side of a river, I measure a line CD along the river (C being nearly opposite A), and observe the angles ACB , $58^\circ 20'$; ACD , $95^\circ 20'$; ADB , $53^\circ 30'$; BDC , $98^\circ 45'$. CD is 600 feet. What is the distance required?

51. Wishing to find the height of a summit A , I measure a horizontal base line CD , 440 yards. At C , the angle of elevation of A is $37^\circ 18'$, and the horizontal angle between D and the summit is $76^\circ 18'$; at D , the horizontal angle between C and the summit is $67^\circ 14'$. Find the height.

52. A balloon is observed from two stations 3000 feet apart. At the first station the horizontal angle of the balloon and the other station is $75^\circ 25'$, and the angle of elevation of the balloon is 18° . The horizontal angle of the first station and the balloon, measured at the second station, is $64^\circ 30'$. Find the height of the balloon.

53. Two forces, one of 410 pounds, and the other of 320 pounds, make an angle of $51^\circ 37'$. Find the intensity and the direction of their resultant.

54. An unknown force, combined with one of 128 pounds, produces a resultant of 200 pounds, and this resultant makes an angle of $18^\circ 24'$ with the known force. Find the intensity and direction of the unknown force.

55. At two stations, the height of a kite subtends the same angle A . The angle which the line joining one station and the kite subtends at the other station is B ; and the distance between the two stations is a . Show that the height of the kite is $\frac{1}{2} a \sin A \sec B$.

56. Two towers on a horizontal plane are 120 feet apart. A person standing successively at their bases observes that the angle of elevation of one is double that of the other; but, when he is half-way between them, the angles of elevation are complementary. Prove that the heights of the towers are 90 and 40 feet.

57. To find the distance of an inaccessible point C from either of two points A and B , having no instruments to measure angles. Prolong CA to a , and CB to b , and join AB , Ab , and Ba . Measure AB , 500; aA , 100; aB , 560; bB , 100; and Ab , 550. Compute the distances AC and BC .

58. Two inaccessible points A and B are visible from D , but no other point can be found whence both are visible. Take some point C , whence A and D can be seen, and measure CD , 200 feet; ADC , 89° ; ACD , $50^\circ 30'$. Then take some point E , whence D and B are visible, and measure DE , 200 feet; BDE , $54^\circ 30'$; BED , $88^\circ 30'$. At D measure ADB , $72^\circ 30'$. Compute the distance AB .

59. To compute the horizontal distance between two inaccessible points A and B , when no point can be found whence both can be seen. Take two points C and D , distant 200 yards, so that A can be seen from C , and B from D . From C measure CF , 200 yards to F , whence A can be seen; and from D measure DE , 200 yards to E , whence B can be seen. Measure AFC , 83° ; ACD , $53^\circ 30'$; ACF , $54^\circ 31'$; BDE , $54^\circ 30'$; BDC , $156^\circ 25'$; DEB , $88^\circ 30'$.

60. A column in the north temperate zone is east-southeast of an observer, and at noon the extremity of its shadow is northeast of him. The shadow is 80 feet in length, and the elevation of the column, at the observer's station, is 45° . Find the height of the column.

61. From the top of a hill the angles of depression of two objects situated in the horizontal plane of the base of the hill are 45° and 30° ; and the horizontal angle between the two objects is 30° . Show that the height of the hill is equal to the distance between the objects.

62. Wishing to know the breadth of a river from A to B , I take AC , 100 yards in the prolongation of BA , and then take CD , 200 yards at right angles to AC . The angle BDA is $37^\circ 18' 30''$. Find AB .

63. The sum of the sides of a triangle is 100. The angle at A is double that at B , and the angle at B is double that at C . Determine the sides.

64. If $\sin^2 A + 5 \cos^2 A = 3$, find A .

65. If $\sin^2 A = m \cos A - n$, find $\cos A$.

66. Given $\sin A = m \sin B$, and $\tan A = n \tan B$; find $\sin A$ and $\cos B$.

67. If $\tan^2 A + 4 \sin^2 A = 6$, find A .

68. If $\sin A = \sin 2A$, find A .

69. If $\tan 2A = 3 \tan A$, find A .

70. Prove that $\tan 50^\circ + \cot 50^\circ = 2 \sec 10^\circ$.

71. Given a regular polygon of n sides, and calling one of them a , find expressions for the radii of the inscribed and the circumscribed circles in terms of n and a .

If P , H , D are the sides of a regular inscribed pentagon, hexagon, and decagon, prove $P^2 = H^2 + D^2$.

AREAS

72. Obtain the formula for the area of a triangle, given two sides b , c , and the included angle A .

73. Obtain the formula for the area of a triangle, given two angles A and B , and included side c .

74. Obtain the formula for the area of a triangle, given the three sides.

75. If a is the side of an equilateral triangle, show that its area is $\frac{a^2\sqrt{3}}{4}$.

76. Two consecutive sides of a rectangle are 52.25 chains and 38.24 chains. Find the area.

77. Two sides of a parallelogram are 59.8 chains and 37.05 chains, and the included angle is $72^\circ 10'$. Find the area.

78. Two sides of a parallelogram are 15.36 chains and 11.46 chains, and the included angle is $47^\circ 30'$. Find the area.

79. Two sides of a triangle are 12.38 chains and 6.78 chains, and the included angle is $46^\circ 24'$. Find the area.

80. Two sides of a triangle are 18.37 chains and 13.44 chains, and they form a right angle. Find the area.

81. Two angles of a triangle are $76^\circ 54'$ and $57^\circ 33' 12''$, and the included side is 9 chains. Find the area.

82. Two sides of a triangle are 19.74 chains and 17.34 chains. The first bears N. $82^\circ 30'$ W.; the second S. $24^\circ 15'$ E. Find the area.

83. The three sides of a triangle are 49 chains, 50.25 chains, and 25.69 chains. Find the area.

84. The three sides of a triangle are 10.64 chains, 12.28 chains, and 9 chains. Find the area.

85. The sides of a triangular field, of which the area is 14 acres, are in the ratio of 3, 5, 7. Find the sides.

86. In the quadrilateral $ABCD$ we have AB , 17.22 chains; AD , 7.45 chains; CD , 14.10 chains; BC , 5.25 chains; and the diagonal AC , 15.04 chains. Find the area.

87. The diagonals of a quadrilateral are a and b , and they intersect at an angle D . Show that the area of the quadrilateral is $\frac{1}{2}ab \sin D$.

88. The diagonals of a quadrilateral are 34 and 56, intersecting at an angle of 67° . Find the area.

89. The diagonals of a quadrilateral are 75 and 49, intersecting at an angle of 42° . Find the area.

90. Show that the area of a regular polygon of n sides, of which one is a , is $\frac{na^2}{4} \cot \frac{180^\circ}{n}$.

91. One side of a regular pentagon is 25. Find the area.

92. One side of a regular hexagon is 32. Find the area.

93. One side of a regular decagon is 46. Find the area.

94. Find the area of a circle whose circumference is 74 feet.

95. Find the area of a circle whose radius is 125 feet.

96. In a circle with a diameter of 125 feet find the area of a sector with an arc of 22° .

97. In a circle with a radius of 44 feet find the area of a sector with an arc of 25° .

98. In a circle with a diameter of 50 feet find the area of a segment with an arc of 280° .

99. Find the area of a segment (less than a semicircle) of which the chord is 20, and the distance of the chord from the middle point of the smaller arc is 2.

100. If r is the radius of a circle, the area of a regular circumscribed polygon of n sides is $nr^2 \tan \frac{180^\circ}{n}$.

The area of a regular inscribed polygon is $\frac{n}{2}r^2 \sin \frac{360^\circ}{n}$.

101. If a is a side of a regular polygon of n sides, the area of the inscribed circle is $\frac{\pi a^2}{4} \cot^2 \frac{180^\circ}{n}$.

The area of the circumscribed circle is $\frac{\pi a^2}{4} \csc^2 \frac{180^\circ}{n}$.

102. The area of a regular polygon inscribed in a circle is to that of the circumscribed regular polygon of the same number of sides as 3 to 4. Find the number of sides.

103. The area of a regular polygon inscribed in a circle is the geometric mean between the areas of an inscribed and a circumscribed regular polygon of half the number of sides.

104. The area of a circumscribed regular polygon is the harmonic mean between the areas of an inscribed regular polygon of the same number of sides and of a circumscribed regular polygon of half that number.

105. The perimeter of a circumscribed regular triangle is double that of the inscribed regular triangle.

106. The square described about a circle is four-thirds the inscribed regular dodecagon.

107. Two sides of a triangle are 3 and 12, and the included angle is 30° . Find the hypotenuse of an isosceles right triangle of equal area.

PLANE SAILING

Plane Sailing is that branch of Navigation in which the surface of the earth is considered a plane. The problems which arise are therefore solved by the methods of Plane Trigonometry.

The *difference of latitude* of two places is the arc of a meridian comprehended between the parallels of latitude passing through those places.

The *departure* between two meridians is the arc of a parallel of latitude comprehended between those meridians. It diminishes as the distance from the equator increases.

When a ship sails in such a manner as to cross successive meridians at the same angle, it is said to sail on a *rhumb-line*. This angle is called the *course*, and the *distance* between two places is measured on a rhumb-line.

If we consider the distance, departure, and difference of latitude of two places to be straight lines, lying in one plane, they form a right triangle, called *the triangle of plane sailing*. If ABC is a plane triangle, right-angled at B , and BC represents the difference of latitude of B and C , ACB will be the course from C to A , CA the distance, and D the departure, measured from B , between the meridian of A and that of B .

108. Taking the earth's equatorial diameter to be 7925.6 miles, find the length in feet of the arc of one minute of a great circle.*

109. A ship sails from latitude $43^{\circ} 45'$ S., on a course N. by E. 2345 miles. Find the latitude reached, and the departure made.

110. A ship sails from latitude $1^{\circ} 45'$ N., on a course S.E. by E., and reaches latitude $2^{\circ} 31'$ S. Find the distance, and the departure.

111. A ship sails from latitude $13^{\circ} 17'$ S., on a course N.E. by E. $\frac{3}{4}$ E., until the departure is 207 miles. Find the distance, and the latitude reached.

112. A ship sails on a course between S. and E. 244 miles, leaving latitude $2^{\circ} 52'$ S., and reaching latitude $5^{\circ} 8'$ S. Find the course, and the departure.

* The length of the arc of one minute of a great circle of the earth is called a *geographical mile* or a *knot*. In the following problems, this is the distance meant by the term "mile," unless otherwise stated.

113. A ship sails from latitude $32^{\circ} 18' \text{ N.}$, on a course between N. and W., a distance of 344 miles, and a departure of 103 miles. Find the course, and the latitude reached.

114. A ship sails on a course between S. and E., making a difference of latitude 136 miles, and a departure 203 miles. Find the distance, and the course.

115. A ship sails due north 15 *statute* miles an hour, for one day. What is the distance, in a straight line, from the point left to the point reached? (Take earth's radius, 3962.8 statute miles.)

PARALLEL AND MIDDLE LATITUDE SAILING

The *difference of longitude* of two places is the angle at the pole made by the meridians of these two places; or, it is the arc of the equator comprehended between these two meridians.

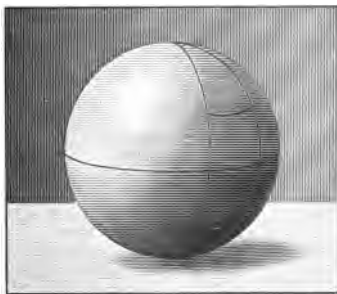


FIG. 68

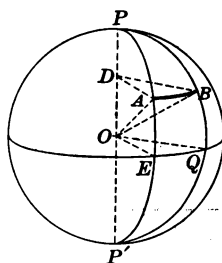


FIG. 69

In **Parallel Sailing** a vessel is supposed to sail due east or due west. The distance sailed is the departure made; and the difference of longitude is found as follows:

116. Given the departure between any two meridians at any latitude; find the difference of longitude of any point on one meridian from any point on the other.

SOLUTION. In rt. $\triangle ODA$, $\angle AOD = 90^\circ - \text{lat.}$

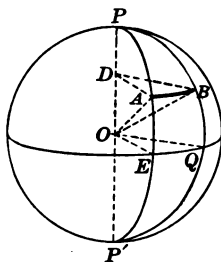


FIG. 70

Hence, $\frac{DA}{OA} = \sin(90^\circ - \text{lat.}) = \cos \text{lat.}$

The $\triangle DAB$ and OEQ are similar.

Therefore, $\frac{DA}{OE} = \frac{AB}{EQ}$, or $\frac{DA}{OA} = \frac{AB}{EQ}$.

Hence, $\cos \text{lat.} = \frac{AB}{EQ}$.

Therefore, $EQ = \frac{AB}{\cos \text{lat.}} = AB \times \sec \text{lat.}$

That is, **Diff. long. = depart. $\times$ sec lat.**

117. A ship in latitude $42^{\circ} 16' \text{ N.}$, longitude $72^{\circ} 16' \text{ W.}$, sails due east a distance of 149 miles. What is the position of the point reached?

118. A ship in latitude $44^{\circ} 49' \text{ S.}$, longitude $119^{\circ} 42' \text{ E.}$, sails due west until it reaches longitude $117^{\circ} 16' \text{ E.}$ Find the distance made.

In **Middle Latitude Sailing** the departure between two places is measured on that parallel of latitude which lies midway between the parallels of the two places. Except in very high latitudes or excessive runs, this assumption produces no great error. Hence, in middle latitude sailing,

Diff. long. = depart. $\times$ sec mid. lat.

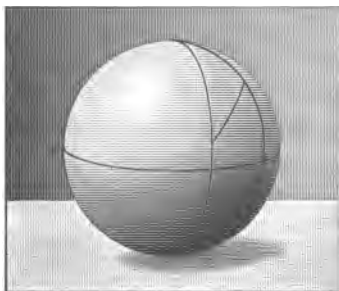


FIG. 71

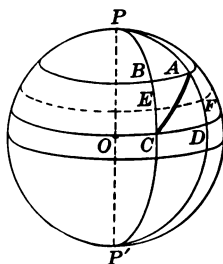


FIG. 72

119. A ship leaves latitude $31^{\circ} 14' N.$, longitude $42^{\circ} 19' W.$, and sails E.N.E. 325 miles. Find the position reached.

120. Find the bearing and distance of Cape Cod from Havana. (Cape Cod $42^{\circ} 2' N.$, $70^{\circ} 3' W.$; Havana, $23^{\circ} 9' N.$, $82^{\circ} 22' W.$)

121. Leaving latitude $49^{\circ} 57' N.$, longitude $15^{\circ} 16' W.$, a ship sails between S. and W. till the departure is 194 miles, and the latitude is $47^{\circ} 18' N.$ Find the course, distance, and longitude reached.

122. Leaving latitude $42^{\circ} 30' N.$, longitude $58^{\circ} 51' W.$, a ship sails S.E. by S. 300 miles. Find the position reached.

123. Leaving latitude $49^{\circ} 57' N.$, longitude $30^{\circ} W.$, a ship sails S. $39^{\circ} W.$, and reaches latitude $47^{\circ} 44' N.$ Find the distance, and longitude reached.

124. Leaving latitude $37^{\circ} N.$, longitude $32^{\circ} 16' W.$, a ship sails between N. and W. 300 miles, and reaches latitude $41^{\circ} N.$ Find the course, and longitude reached.

125. Leaving latitude $50^{\circ} 10' S.$, longitude $30^{\circ} E.$, a ship sails E.S.E., making a departure of 160 miles. Find the distance, and position reached.

126. Leaving latitude $49^{\circ} 30' N.$, longitude $25^{\circ} W.$, a ship sails between S. and E. 215 miles, making a departure of 167 miles. Find the course, and position reached.

127. Leaving latitude $43^{\circ} S.$, longitude $21^{\circ} W.$, a ship sails 273 miles, and reaches latitude $40^{\circ} 17' S.$ What are the *two* courses and longitudes which will satisfy the data?

128. Leaving latitude $17^{\circ} N.$, longitude $119^{\circ} E.$, a ship sails 219 miles, making a departure of 162 miles. What four sets of answers do we get?

129. A ship in latitude 30° sails due east 360 statute miles. What is the shortest distance from the point left to the point reached? Solve the same problem for latitude 45° , 60° .

TRAVERSE SAILING

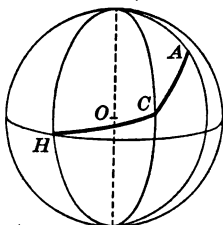


FIG. 73

Traverse Sailing is the application of the principles of Plane and Middle Latitude Sailing to cases when the ship sails from one point to another on two or more different courses. Each course is worked by itself, and these independent results are combined, as may be seen in the solution of the following examples.

130. Leaving latitude $37^{\circ} 16' S.$, longitude $18^{\circ} 42' W.$, a ship sails N.E. 104 miles, then N.N.W. 60 miles, then W. by S. 216 miles. Find the position reached, and its bearing and distance from the point left.

We have, for the first course, difference of latitude 73.5 N., departure 73.5 E.; for the second course, difference of latitude 55.4 N., departure 23 W.; for the third course, difference of latitude 42.1 S., departure 211.8 W.

On the whole, then, the ship has made 128.9 miles of north latitude, and 42.1 miles of south latitude. The place reached is therefore on a parallel of latitude 86.8 miles to the north of the parallel left, that is, in latitude $35^{\circ} 49.2' S.$

The departure is, in the same way, found to be 161.3 miles W.; and the middle latitude is $36^{\circ} 32.6'$. With these data, and the formula after Example 118, we find the difference of longitude to be 201', or $3^{\circ} 21' W.$ Hence, the longitude reached is $22^{\circ} 3' W.$

With the difference of latitude 86.8 miles, and the departure 161.3 miles, we find the course to be N. $61^{\circ} 43' W.$, and the distance 183.2 miles. The ship has reached the same point that it would have reached if it had sailed directly on a course N. $61^{\circ} 43' W.$ for a distance of 183.2 miles.

131. A ship leaves Cape Cod (Example 120), and sails S.E. by S. 114 miles, N. by E. 94 miles, W.N.W. 42 miles. Solve as in Example 130.

132. A ship leaves Cape of Good Hope (latitude $34^{\circ} 22' S.$, longitude $18^{\circ} 30' E.$), and sails N.W. 126 miles, N. by E. 84 miles, W.S.W. 217 miles. Solve as in Example 130.

EXERCISE XXIV

PROBLEMS IN GONIOMETRY

Prove that:

1. $\sin x + \cos x = \sqrt{2} \cos (x - \frac{1}{2} \pi).$
2. $\sin x - \cos x = -\sqrt{2} \cos (x + \frac{1}{2} \pi).$
3. $\sin x + \sqrt{3} \cos x = 2 \sin (x + \frac{1}{3} \pi).$
4. $\sin (x + \frac{1}{3} \pi) + \sin (x - \frac{1}{3} \pi) = \sin x.$
5. $\cos (x + \frac{1}{3} \pi) + \cos (x - \frac{1}{3} \pi) = \sqrt{3} \cos x.$
6. $\tan x + \sec x = \tan (\frac{1}{2} x + \frac{1}{2} \pi).$
7. $\tan x + \sec x = \frac{1}{\sec x - \tan x}.$
8. $\frac{1 - \tan x}{1 + \tan x} = \frac{\cot x - 1}{\cot x + 1}.$
9. $\frac{\sin x}{1 + \cos x} + \frac{1 + \cos x}{\sin x} = 2 \csc x.$
10. $\tan x + \cot x = 2 \csc 2x.$
11. $\cot x - \tan x = 2 \cot 2x.$
12. $1 + \tan x \tan 2x = \sec 2x.$
13. $\sec 2x = \frac{\sec^2 x}{2 - \sec^2 x}.$
14. $2 \sec 2x = \sec (x + 45^{\circ}) \sec (x - 45^{\circ}).$

$$15. \tan 2x + \sec 2x = \frac{\cos x + \sin x}{\cos x - \sin x}.$$

$$16. \sin 2x = \frac{2 \tan x}{1 + \tan^2 x}.$$

$$17. 2 \sin x + \sin 2x = \frac{2 \sin^3 x}{1 - \cos x}.$$

$$18. \sin 3x = \frac{\sin^3 2x - \sin^2 x}{\sin x}.$$

$$19. \tan 3x = \frac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}.$$

$$20. \frac{\tan 2x + \tan x}{\tan 2x - \tan x} = \frac{\sin 3x}{\sin x}.$$

$$21. \sin(x+y) + \cos(x-y) = 2 \sin(x + \frac{1}{2}\pi) \sin(y + \frac{1}{2}\pi).$$

$$22. \sin(x+y) - \cos(x-y) = -2 \sin(x - \frac{1}{2}\pi) \sin(y - \frac{1}{2}\pi).$$

$$23. \tan x + \tan y = \frac{\sin(x+y)}{\cos x \cos y}.$$

$$24. \tan(x+y) = \frac{\sin 2x + \sin 2y}{\cos 2x + \cos 2y}.$$

$$25. \frac{\sin x + \cos y}{\sin x - \cos y} = \frac{\tan \{ \frac{1}{2}(x+y) + 45^\circ \}}{\tan \{ \frac{1}{2}(x-y) - 45^\circ \}}.$$

$$26. \sin 2x + \sin 4x = 2 \sin 3x \cos x.$$

$$27. \sin 4x = 4 \sin x \cos x - 8 \sin^3 x \cos x \\ = 8 \cos^3 x \sin x - 4 \cos x \sin x.$$

$$28. \cos 4x = 1 - 8 \cos^2 x + 8 \cos^4 x = 1 - 8 \sin^2 x + 8 \sin^4 x.$$

$$29. \cos 2x + \cos 4x = 2 \cos 3x \cos x.$$

$$30. \sin 3x - \sin x = 2 \cos 2x \sin x.$$

$$31. \sin^3 x \sin 3x + \cos^3 x \cos 3x = \cos^2 2x.$$

$$32. \cos^4 x - \sin^4 x = \cos 2x.$$

$$33. \cos^4 x + \sin^4 x = 1 - \frac{1}{2} \sin^2 2x.$$

$$34. \cos^6 x - \sin^6 x = (1 - \sin^2 x \cos^2 x) \cos 2x.$$

$$35. \cos^6 x + \sin^6 x = 1 - 3 \sin^2 x \cos^2 x.$$

$$36. \frac{\sin 3x + \sin 5x}{\cos 3x - \cos 5x} = \cot x.$$

$$37. \frac{\sin 3x + \sin 5x}{\sin x + \sin 3x} = 2 \cos 2x.$$

$$38. \csc x - 2 \cot 2x \cos x = 2 \sin x.$$

$$39. (\sin 2x - \sin 2y) \tan(x + y) = 2(\sin^2 x - \sin^2 y).$$

$$40. (1 + \cot x + \tan x)(\sin x - \cos x) = \frac{\sec x}{\csc^2 x} - \frac{\csc x}{\sec^2 x}.$$

$$41. \sin x + \sin 3x + \sin 5x = \frac{\sin^2 3x}{\sin x}.$$

$$42. \frac{3 \cos x + \cos 3x}{3 \sin x - \sin 3x} = \cot^3 x.$$

$$43. \sin 3x = 4 \sin x \sin(60^\circ + x) \sin(60^\circ - x).$$

$$44. \sin 4x = 2 \sin x \cos 3x + \sin 2x.$$

$$45. \sin x + \sin(x - \frac{2}{3}\pi) + \sin(\frac{1}{3}\pi - x) = 0.$$

$$46. \cos x \sin(y - z) + \cos y \sin(z - x) + \cos z \sin(x - y) = 0.$$

$$47. \begin{aligned} \cos(x + y) \sin y - \cos(x + z) \sin z \\ = \sin(x + y) \cos y - \sin(x + z) \cos z. \end{aligned}$$

$$48. \begin{aligned} \cos(x + y + z) + \cos(x + y - z) + \cos(x - y + z) \\ + \cos(y + z - x) = 4 \cos x \cos y \cos z. \end{aligned}$$

$$49. \begin{aligned} \sin(x + y) \cos(x - y) + \sin(y + z) \cos(y - z) \\ + \sin(z + x) \cos(z - x) = \sin 2x + \sin 2y + \sin 2z. \end{aligned}$$

$$50. \frac{\sin 75^\circ + \sin 15^\circ}{\sin 75^\circ - \sin 15^\circ} = \tan 60^\circ.$$

$$51. \cos 20^\circ + \cos 100^\circ + \cos 140^\circ = 0.$$

$$52. \cos 36^\circ + \sin 36^\circ = \sqrt{2} \cos 9^\circ.$$

$$53. \tan 11^\circ 15' + 2 \tan 22^\circ 30' + 4 \tan 45^\circ = \cot 11^\circ 15'.$$

If A, B, C are the angles of a plane triangle, prove that:

$$54. \sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C.$$

$$55. \cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C.$$

$$56. \sin 3A + \sin 3B + \sin 3C = -4 \cos \frac{3A}{2} \cos \frac{3B}{2} \cos \frac{3C}{2}.$$

$$57. \cos^2 A + \cos^2 B + \cos^2 C = 1 - 2 \cos A \cos B \cos C.$$

If $A + B + C = 90^\circ$, prove that:

$$58. \tan A \tan B + \tan B \tan C + \tan C \tan A = 1.$$

$$59. \sin^2 A + \sin^2 B + \sin^2 C = 1 - 2 \sin A \sin B \sin C.$$

$$60. \sin 2A + \sin 2B + \sin 2C = 4 \cos A \cos B \cos C.$$

Prove that:

$$61. \sin(\sin^{-1}x + \sin^{-1}y) = x\sqrt{1-y^2} + y\sqrt{1-x^2}.$$

$$62. \tan(\tan^{-1}x + \tan^{-1}y) = \frac{x+y}{1-xy}.$$

$$63. 2 \tan^{-1}x = \tan^{-1} \frac{2x}{1-x^2}.$$

$$64. 2 \sin^{-1}x = \sin^{-1}(2x\sqrt{1-x^2}).$$

$$65. 2 \cos^{-1}x = \cos^{-1}(2x^2 - 1).$$

$$66. 3 \tan^{-1}x = \tan^{-1} \frac{3x - x^3}{1 - 3x^2}.$$

$$67. \sin^{-1} \sqrt{\frac{x}{y}} = \tan^{-1} \sqrt{\frac{x}{y-x}}.$$

$$68. \sin^{-1} \sqrt{\frac{x-y}{x-z}} = \tan^{-1} \sqrt{\frac{x-y}{y-z}}.$$

69. $\sin^{-1}x = \sec^{-1} \frac{1}{\sqrt{1-x^2}}$.
70. $2 \sec^{-1}x = \tan^{-1} \frac{2\sqrt{x^2-1}}{2-x^2}$.
71. $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = 45^\circ$.
72. $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{4} = \tan^{-1}\frac{7}{4}$.
73. $\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{1}{5} = \sin^{-1}\frac{4}{5}$.
74. $\sin^{-1} \frac{1}{\sqrt{82}} + \sin^{-1} \frac{4}{\sqrt{41}} = 45^\circ$.
75. $\sec^{-1}\frac{5}{3} + \sec^{-1}\frac{1}{3} = 75^\circ 45'$.
76. $\tan^{-1}(2 + \sqrt{3}) - \tan^{-1}(2 - \sqrt{3}) = \sec^{-1}2$.
77. $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{6} = 45^\circ$.
78. $\tan^{-1} \frac{1}{1-2x+4x^2} + \tan^{-1} \frac{1}{1+2x+4x^2} = \tan^{-1} \frac{1}{2x^2}$.
79. Given $\cos x = \frac{3}{4}$; find $\sin \frac{1}{2}x$ and $\cos \frac{1}{2}x$.
80. Given $\tan x = \frac{1}{2}$; find $\tan \frac{1}{2}x$.
81. Given $\sin x + \cos x = \sqrt{\frac{1}{2}}$; find $\cos 2x$.
82. Given $\tan 2x = 2\frac{1}{4}$; find $\sin x$.
83. Given $\cos 3x = \frac{3}{4}$; find $\tan x$.
84. Given $2 \csc x - \cot x = \sqrt{3}$; find $\sin \frac{1}{2}x$.
85. Find $\sin 18^\circ$ and $\cos 36^\circ$.

Find the value of:

86. $a \sec x + b \csc x$, when $\tan x = \sqrt[3]{\frac{b}{a}}$.
87. $\sin 3x$, when $\sin 2x = \sqrt{1-m^2}$.
88. $\sin x$, when $\tan^2 x + 3 \cot^2 x = 4$.

$$89. \frac{\csc^2 x - \sec^2 x}{\csc^2 x + \sec^2 x}, \text{ when } \tan x = \sqrt{\frac{1}{2}}.$$

$$90. \cos x, \text{ when } 5 \tan x + \sec x = 5.$$

$$91. \sec x, \text{ when } \tan x = \frac{a}{\sqrt{2a+1}}.$$

Simplify the following expressions :

$$92. \frac{(\cos x + \cos y)^2 + (\sin x + \sin y)^2}{\cos^2 \frac{1}{2}(x-y)}.$$

$$93. \frac{\sin(x+2y) - 2\sin(x+y) + \sin x}{\cos(x+2y) - 2\cos(x+y) + \cos x}.$$

$$94. \frac{\sin(x-z) + 2\sin x + \sin(x+z)}{\sin(y-z) + 2\sin y + \sin(y+z)}.$$

$$95. \frac{\cos 6x - \cos 4x}{\sin 6x + \sin 4x}.$$

$$96. \tan^{-1}(2x+1) + \tan^{-1}(2x-1).$$

$$97. \frac{1}{1+\sin^2 x} + \frac{1}{1+\cos^2 x} + \frac{1}{1+\sec^2 x} + \frac{1}{1+\csc^2 x}.$$

$$98. 2\sec^2 x - \sec^4 x - 2\csc^2 x + \csc^4 x.$$

SOLUTION OF SINGLE EQUATIONS

To solve a single equation that involves different functions of the same angle, or the same or different functions of related angles, first transform the equation, if necessary, into an equivalent equation that involves a single function of the same angle.

Employ the method of factoring, if possible, in the algebraic part of the solution.

Completely solve each equation, and check the results by substitution in the given equation.

Solve $\cos x = \sin 2x$.

By [12], p. 58,

$$\sin 2x = 2 \sin x \cos x.$$

$$\therefore \cos x = 2 \sin x \cos x.$$

$$\therefore (1 - 2 \sin x) \cos x = 0.$$

$$\therefore \cos x = 0, \text{ or } 1 - 2 \sin x = 0.$$

$$\therefore x = 90^\circ \text{ or } 270^\circ, \text{ or } 30^\circ \text{ or } 150^\circ.$$

Each of these values satisfies the given equation.

Solve the following equations :

- | | |
|---|--|
| 99. $\sin x = 2 \sin (\frac{1}{3}\pi + x).$ | 111. $\sin x = \cos 2x.$ |
| 100. $\sin 2x = 2 \cos x.$ | 112. $\tan x \tan 2x = 2.$ |
| 101. $\cos 2x = 2 \sin x.$ | 113. $\sec x = 4 \csc x.$ |
| 102. $\sin x + \cos x = 1.$ | 114. $\cos \theta + \cos 2\theta = 0.$ |
| 103. $\sin x + \cos 2x = 4 \sin^2 x.$ | 115. $\cot \frac{1}{2}\theta + \csc \theta = 2.$ |
| 104. $4 \cos 2x + 3 \cos x = 1.$ | 116. $\cot x \tan 2x = 3.$ |
| 105. $\sin x + \sin 2x = \sin 3x.$ | 117. $\sin x \sec 2x = 1.$ |
| 106. $\sin 2x = 3 \sin^2 x - \cos^2 x.$ | 118. $\sin^2 x + \sin 2x = 1.$ |
| 107. $\cot \theta = \frac{1}{3} \tan \theta.$ | 119. $\cos x \sin 2x \csc x = 1.$ |
| 108. $2 \sin \theta = \cos \theta.$ | 120. $\cot x \tan 2x = \sec 2x.$ |
| 109. $2 \sin^2 x + 5 \sin x = 3.$ | 121. $\sin 2x = \cos 4x.$ |
| 110. $\tan x \sec x = \sqrt{2}.$ | 122. $\sin 2z \cot z - \sin^2 z = \frac{1}{2}.$ |
| 123. $\tan x + \tan 2x = \tan 3x.$ | |
| 124. $\cot x - \tan x = \sin x + \cos x.$ | |
| 125. $\tan^2 x = \sin 2x.$ | |
| 126. $\tan x + \cot x = \tan 2x.$ | |
| 127. $\frac{1 - \tan x}{1 + \tan x} = \cos 2x.$ | |

128. $\sin x + \sin 2x = 1 - \cos 2x$.
129. $\sec 2x + 1 = 2 \cos x$.
130. $\tan 2x + \tan 3x = 0$.
131. $\tan(\frac{1}{4}\pi + x) + \tan(\frac{1}{4}\pi - x) = 4$.
132. $\sqrt{1 + \sin x} - \sqrt{1 - \sin x} = 2 \cos x$.
133. $\tan x \tan 3x = -\frac{2}{3}$.
134. $\sin(45^\circ + x) + \cos(45^\circ - x) = 1$.
135. $\tan x + \sec x = a$.
136. $\cos 2x = a(1 - \cos x)$.
137. $(1 - \tan x) \cos 2x = a(1 + \tan x)$.
138. $\sin^6 x + \cos^6 x = \frac{7}{15} \sin^2 2x$.
139. $\cos 3x + 8 \cos^3 x = 0$.
140. $\sec(x + 120^\circ) + \sec(x - 120^\circ) = 2 \cos x$.
141. $\csc x = \cot x + \sqrt{3}$.
142. $4 \cos 2x + 6 \sin x = 5$
143. $\cos x - \cos 2x = 1$.
144. $\sin 4x - \sin 2x = \sin x$.
145. $2 \sin^2 x + \sin^2 2x = 2$.
146. $\cos 5x + \cos 3x + \cos x = 0$.
147. $\sec x - \cot x = \csc x - \tan x$.
148. $\tan^2 x + \cot^2 x = \frac{10}{3}$.
149. $\sin 4x - \cos 3x = \sin 2x$.
150. $\sin x + \cos x = \sec x$.
151. $2 \cos x \cos 3x + 1 = 0$.
152. $\cos 3x - 2 \cos 2x + \cos x = 0$.

153. $\tan 2x \tan x = 1$.

154. $\sin(x + 12^\circ) + \sin(x - 8^\circ) = \sin 20^\circ$.

155. $\tan(60^\circ + x) \tan(60^\circ - x) = -2$.

156. $\sin(x + 120^\circ) + \sin(x + 60^\circ) = \frac{3}{2}$.

157. $\sin(x + 30^\circ) \sin(x - 30^\circ) = \frac{1}{2}$.

158. $\sin^4 x + \cos^4 x = \frac{5}{8}$.

159. $\sin^4 x - \cos^4 x = \frac{7}{8}$.

160. $\tan(x + 30^\circ) = 2 \cos x$.

161. $\sec x = 2 \tan x + \frac{1}{2}$.

162. $\sin 11x \sin 4x + \sin 5x \sin 2x = 0$.

163. $\cos x + \cos 3x + \cos 5x + \cos 7x = 0$.

164. $\sin(x + 12^\circ) \cos(x - 12^\circ) = \cos 33^\circ \sin 57^\circ$.

165. $\sin^{-1} x + \sin^{-1} \frac{1}{2} x = 120^\circ$.

166. $\tan^{-1} x + \tan^{-1} 2x = \tan^{-1} 3\sqrt{3}$.

167. $\sin^{-1} x + 2 \cos^{-1} x = \frac{3}{4} \pi$.

168. $\sin^{-1} x + 3 \cos^{-1} x = 210^\circ$.

169. $\tan^{-1} x + 2 \cot^{-1} x = 135^\circ$.

170. $\tan^{-1}(x + 1) + \tan^{-1}(x - 1) = \tan^{-1} 2x$.

171. $\tan^{-1} \frac{x+2}{x+1} + \tan^{-1} \frac{x-2}{x-1} = \frac{3}{4} \pi$.

172. $\tan^{-1} \frac{2x}{1-x^2} = 60^\circ$.

173. $\cos 2\theta \sec \theta + \sec \theta + 1 = 0$.

174. $\sin x \cos 2x \tan x \cot 2x \sec x \csc 2x = 1$.

175. $\sin \frac{1}{2} x (\cos 2x - 2) (1 - \tan^2 x) = 0$.

HINT. Equate to 0 each factor except the second. The second factor cannot equal 0.

176. $\sin 3x = \cos 2x - 1$. 178. $\sin 2\theta = \cos 3\theta$.
 177. $\tan x + \tan 2x = 0$. 179. $(3 - 4 \cos^2 x) \sin 2x = 0$.
 180. $\sin x + \sin 2x + \sin 3x = 0$.
 181. $\sin \theta + 2 \sin 2\theta + 3 \sin 3\theta = 0$.
 182. $\sin^2 x \cos^2 x - \cos^2 x - \sin^2 x + 1 = 0$.
 183. $\sin x + \sin 3x = \cos x - \cos 3x$.
 184. $(1 - \sqrt{1 - \tan^2 x}) \cos 2x \text{ vers } 3x = 0$.
 185. $\tan(\theta + 45^\circ) = 8 \tan \theta$.
 186. $\sin(x - 30^\circ) = \frac{1}{2} \sqrt{3} \sin x$.
 187. $\tan(\theta + 45^\circ) \tan \theta = 2$.
 188. $\sin^{-1} \frac{1}{2} x = 30^\circ$.

SYSTEMS OF EQUATIONS

189. Solve for x and y the system

$$x \sin \alpha + y \sin \beta = a, \quad (1)$$

$$x \cos \alpha + y \cos \beta = b. \quad (2)$$

$$(1) \times \cos \alpha, \quad x \sin \alpha \cos \alpha + y \sin \beta \cos \alpha = a \cos \alpha. \quad (3)$$

$$(2) \times \sin \alpha, \quad x \sin \alpha \cos \alpha + y \cos \beta \sin \alpha = b \sin \alpha. \quad (4)$$

$$(3) - (4), \quad y(\sin \beta \cos \alpha - \cos \beta \sin \alpha) = a \cos \alpha - b \sin \alpha. \quad (5)$$

$$\therefore y = \frac{a \cos \alpha - b \sin \alpha}{\sin(\beta - \alpha)}.$$

Similarly,
$$x = \frac{b \sin \beta - a \cos \beta}{\sin(\beta - \alpha)}.$$

190. Solve for x and y the system

$$\sin x + \sin y = a, \quad (1)$$

$$\cos x + \cos y = b. \quad (2)$$

Transform (1) and (2), by Sect. XXXII,

$$\text{by [20], p. 59,} \quad 2 \sin \frac{1}{2}(x+y) \cos \frac{1}{2}(x-y) = a, \quad (3)$$

$$\text{by [22], p. 59,} \quad 2 \cos \frac{1}{2}(x+y) \cos \frac{1}{2}(x-y) = b. \quad (4)$$

$$(3) \div (4), \quad \tan \frac{1}{2}(x+y) = \frac{a}{b}. \quad (5)$$

$$\therefore \sin \frac{1}{2}(x+y) = \frac{a}{\sqrt{a^2 + b^2}}. \quad (6)$$

Substitute value of $\sin \frac{1}{2}(x+y)$ in (3),

$$\cos \frac{1}{2}(x-y) = \frac{1}{2} \sqrt{a^2 + b^2}. \quad (7)$$

$$\text{From (5),} \quad x+y = 2 \tan^{-1} \frac{a}{b}. \quad (8)$$

$$\text{From (7),} \quad x-y = 2 \cos^{-1} \frac{1}{2} \sqrt{a^2 + b^2}. \quad (9)$$

$$\text{Whence} \quad x = \tan^{-1} \frac{a}{b} + \cos^{-1} \frac{1}{2} \sqrt{a^2 + b^2},$$

$$\text{and} \quad y = \tan^{-1} \frac{a}{b} - \cos^{-1} \frac{1}{2} \sqrt{a^2 + b^2}.$$

191. Solve for r and θ the system

$$r \sin \theta = a, \quad (1)$$

$$r \cos \theta = b. \quad (2)$$

$$(1) \div (2), \quad \tan \theta = \frac{a}{b}. \quad (3)$$

$$\text{From (3),} \quad \theta = \tan^{-1} \frac{a}{b}. \quad (4)$$

Square (1) and (2) and add,

$$r^2 (\sin^2 \theta + \cos^2 \theta) = a^2 + b^2.$$

$$\therefore r = \sqrt{a^2 + b^2}.$$

192. Solve for r and θ the system

$$r \sin (\theta + \alpha) = a, \quad (1)$$

$$r \cos (\theta + \beta) = b. \quad (2)$$

Expand (1) and (2),

$$r \sin \theta \cos \alpha + r \cos \theta \sin \alpha = a. \quad (3)$$

$$r \cos \theta \cos \beta - r \sin \theta \sin \beta = b. \quad (4)$$

Now solve (3) and (4) for $r \sin \theta$ and $r \cos \theta$, as in Example 189. Then solve for r and θ , as in Example 191.

193. Solve for r , θ , and ϕ the system

$$r \cos \phi \sin \theta = a, \quad (1)$$

$$r \cos \phi \cos \theta = b, \quad (2)$$

$$r \sin \phi = c. \quad (3)$$

$$(1) \div (2), \quad \tan \theta = \frac{a}{b}. \quad \therefore \theta = \tan^{-1} \frac{a}{b}. \quad (4)$$

Square (1) and (2) and add,

$$r^2 \cos^2 \phi = a^2 + b^2. \quad (5)$$

$$(3) \div (5), \quad \tan \phi = \frac{c}{\sqrt{a^2 + b^2}}. \quad \therefore \phi = \tan^{-1} \frac{c}{\sqrt{a^2 + b^2}}. \quad (6)$$

Square (3) and add to (5),

$$r^2 = a^2 + b^2 + c^2. \\ \therefore r = \sqrt{a^2 + b^2 + c^2}. \quad (7)$$

Solve the following systems for r , θ , ϕ , x , and y :

$$194. \quad \begin{aligned} x \sin 21^\circ + y \cos 44^\circ &= 179.70, \\ x \cos 21^\circ + y \sin 44^\circ &= 232.30. \end{aligned}$$

$$195. \quad \begin{aligned} \sin x - \sin y &= 0.7038, \\ \cos x - \cos y &= -0.7245. \end{aligned}$$

$$196. \quad \begin{aligned} r \sin \theta &= 92.344, \\ r \cos \theta &= 205.309. \end{aligned}$$

$$197. \quad \begin{aligned} r \sin (\theta - 19^\circ 18') &= 59.4034, \\ r \cos (\theta - 30^\circ 54') &= 147.9347. \end{aligned}$$

$$198. \quad \begin{aligned} r \cos \phi \cos \theta &= -46.7654, \\ r \sin \phi \cos \theta &= 81, \\ r \sin \theta &= -54. \end{aligned}$$

Find θ from the system

$$190. \quad \begin{aligned} &= r(\theta - \sin \theta), \\ &= r(1 - \cos \theta). \end{aligned}$$

$$= \text{vers}^{-1} \frac{y}{r}.$$

CHAPTER VI

CONSTRUCTION OF TABLES

SECTION XLIII

LOGARITHMS

Properties of Logarithms. Any positive number except unity being selected as a *base*, the index or exponent which the base must have to produce a given number is the logarithm of that number to the given base.

Thus, if $a^n = N$, then $n = \log_a N$.

$n = \log_a N$ is read, n is equal to $\log N$ to the base a .

Let a be the base, M and N any positive numbers, m and n their logarithms to the base a ; so that

$$\begin{aligned} a^m &= M, & a^n &= N, \\ m &= \log_a M, & n &= \log_a N. \end{aligned}$$

Then, in any system of logarithms:

1. *The logarithm of 1 is 0.*

For, $a^0 = 1. \quad \therefore 0 = \log_a 1.$

2. *The logarithm of the base itself is 1.*

For, $a^1 = a. \quad \therefore 1 = \log_a a.$

3. *The logarithm of the reciprocal of a positive number is the negative of the logarithm of the number.*

For, if $a^n = N$, then $\frac{1}{N} = \frac{1}{a^n} = a^{-n}.$

$$\therefore \log_a \left(\frac{1}{N} \right) = -n = -\log_a N.$$

4. *The logarithm of the product of two or more positive numbers is found by adding together the logarithms of the several factors.*

$$\text{For,} \quad M \times N = a^m \times a^n = a^{m+n}.$$

$$\therefore \log_a(M \times N) = m + n = \log_a M + \log_a N.$$

Similarly for the product of three or more factors.

5. *The logarithm of the quotient of two positive numbers is found by subtracting the logarithm of the divisor from the logarithm of the dividend.*

$$\text{For,} \quad \frac{M}{N} = \frac{a^m}{a^n} = a^{m-n}.$$

$$\therefore \log_a \left(\frac{M}{N} \right) = m - n = \log_a M - \log_a N.$$

6. *The logarithm of a power of a positive number is found by multiplying the logarithm of the number by the exponent of the power.*

$$\text{For,} \quad N^p = (a^n)^p = a^{np}.$$

$$\therefore \log_a(N^p) = np = p \log_a N.$$

7. *The logarithm of the real positive value of a root of a positive number is found by dividing the logarithm of the number by the index of the root.*

$$\text{For,} \quad \sqrt[r]{N} = \sqrt[r]{a^n} = a^{\frac{n}{r}}.$$

$$\therefore \log_a \sqrt[r]{N} = \frac{n}{r} = \frac{\log_a N}{r}.$$

Change of System. Logarithms to any base a may be converted into logarithms to any other base b as follows:

Let N be any number, and let

$$n = \log_a N \text{ and } m = \log_b N.$$

Then,

$$N = a^n \text{ and } N = b^m.$$

$$\therefore a^n = b^m.$$

Taking logarithms to any base whatever,

$$n \log a = m \log b,$$

or, $\log a \times \log_b N = \log b \times \log_b N,$

from which $\log_b N$ may be found when $\log a$, $\log b$, and $\log_a N$ are given; and conversely, $\log_a N$ may be found when $\log a$, $\log b$, and $\log_b N$ are given.

Two Important Systems. Although the number of different systems of logarithms is unlimited, there are but two systems which are in common use. These are:

1. The common system, also called the Briggs, denary, or decimal system, of which the base is 10.

2. The natural system of which the base is the fixed value which the sum of the series

$$1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \dots$$

approaches as the number of terms is indefinitely increased. This fixed value, correct to seven places of decimals, is 2.7182818, and is denoted by the letter e .

The common system is used in actual calculation; the natural system is used in the higher mathematics.

EXERCISE XXV

1. Given $\log_{10} 2 = 0.30103$, $\log_{10} 3 = 0.47712$, $\log_{10} 7 = 0.84510$; find $\log_{10} 6$, $\log_{10} 14$, $\log_{10} 21$, $\log_{10} 4$, $\log_{10} 12$, $\log_{10} 5$, $\log_{10} \frac{1}{2}$, $\log_{10} \frac{1}{4}$, $\log_{10} \frac{1}{5}$, $\log_{10} \frac{2}{3}$.

2. With the data of Example 1, find $\log_2 10$, $\log_2 5$, $\log_3 5$, $\log_7 \frac{1}{2}$, $\log_{5\frac{2}{3}} \frac{2}{3}$.

3. Given $\log_{10} e = 0.43429$; find $\log_e 2$, $\log_e 3$, $\log_e 5$, $\log_e 7$, $\log_e 8$, $\log_e 9$, $\log_e \frac{2}{3}$, $\log_e \frac{3}{2}$, $\log_e \frac{5}{2}$, $\log_e \frac{7}{5}$.

4. Find x from the equations $5^x = 12$, $16^x = 10$, $27^x = 4$.

SECTION XLIV

EXPONENTIAL AND LOGARITHMIC SERIES

Exponential Series. By the binomial theorem,

$$\begin{aligned} \left(1 + \frac{1}{n}\right)^{nx} &= 1 + nx \times \frac{1}{n} + \frac{nx(nx-1)}{1 \cdot 2} \times \frac{1}{n^2} \\ &\quad + \frac{nx(nx-1)(nx-2)}{1 \cdot 2 \cdot 3} \times \frac{1}{n^3} + \dots \\ &= 1 + x + \frac{x\left(x - \frac{1}{n}\right)}{[2]} + \frac{x\left(x - \frac{1}{n}\right)\left(x - \frac{2}{n}\right)}{[3]} + \dots \quad (1) \end{aligned}$$

This equation is true for all real values of x , since the binomial theorem may be extended to the case of incommensurable exponents (Wentworth's *College Algebra*, § 299); it is, however, true only for values of n numerically greater than 1, since $\frac{1}{n}$ must be numerically less than 1 (*College Algebra*, § 418).

As (1) is true for all values of x , it is true when $x = 1$.

$$\therefore \left(1 + \frac{1}{n}\right)^n = 1 + 1 + \frac{1 - \frac{1}{n}}{[2]} + \frac{\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)}{[3]} + \dots \quad (2)$$

But $\left[\left(1 + \frac{1}{n}\right)^n\right]^x = \left(1 + \frac{1}{n}\right)^{nx}$.

Hence, from (1) and (2),

$$\begin{aligned} &\left[1 + 1 + \frac{1 - \frac{1}{n}}{[2]} + \frac{\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)}{[3]} + \dots\right]^x \\ &= 1 + x + \frac{x\left(x - \frac{1}{n}\right)}{[2]} + \frac{x\left(x - \frac{1}{n}\right)\left(x - \frac{2}{n}\right)}{[3]} + \dots \end{aligned}$$

This last equation is true for all values of n numerically greater than 1. Taking the limits of the two members as n increases without limit, we obtain

$$\left(1 + 1 + \frac{1}{2} + \frac{1}{3} + \dots\right)^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \dots, \quad (3)$$

and this is true for all values of x . It is easily seen that both series are convergent for all values of x .

The sum of the infinite series in parenthesis is the natural base e .

$$\text{Hence, by (3), } e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \dots \quad (4)$$

To calculate the value of e , we proceed as follows:

	1.000000
2	1.000000
3	0.500000
4	0.166667
5	0.041667
6	0.008333
7	0.001388
8	0.000198
9	0.000025
	0.000003

Adding, $e = 2.71828$.

To ten places, $e = 2.7182818284$.

Limit of $\left(1 + \frac{x}{n}\right)^n$. By the binomial theorem,

$$\begin{aligned} \left(1 + \frac{x}{n}\right)^n &= 1 + n \times \frac{x}{n} + \frac{n(n-1)}{1 \cdot 2} \times \frac{x^2}{n^2} \\ &\quad + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} \times \frac{x^3}{n^3} + \dots \\ &= 1 + x + \frac{1 - \frac{1}{n}}{2} x^2 + \frac{\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)}{3} x^3 + \dots \end{aligned}$$

This equation is true for all values of n greater than x (*College Algebra*, § 418). Take the limit as n increases without limit, x remaining finite; then

$$\begin{aligned}\lim_{n \doteq \infty} \left(1 + \frac{x}{n}\right)^n &= 1 + x + \frac{x^2}{\underline{2}} + \frac{x^3}{\underline{3}} + \dots \\ &= e^x = \lim_{n \doteq \infty} \left(1 + \frac{1}{n}\right)^{nx}.\end{aligned}\quad (5)$$

Logarithmic Series.

Let $y = \log_e(1 + x)$;

then $1 + x = e^y = \lim_{n \doteq \infty} \left(1 + \frac{y}{n}\right)^n$.

If n is merely a large number, but not infinite,

$$\left(1 + \frac{y}{n}\right)^n = 1 + x + \epsilon,$$

where ϵ is a variable number which approaches the limit 0, when n increases without limit. Hence,

$$1 + \frac{y}{n} = \sqrt[n]{1 + x + \epsilon},$$

$$y = n \sqrt[n]{1 + x + \epsilon} - n.$$

If n increases without limit, and consequently ϵ approaches 0 as a limit, we have

$$y = \lim_{n \doteq \infty} [n \sqrt[n]{1 + x} - n].$$

If x is less than 1, we may expand the right-hand member of this equation by the binomial theorem. The result is

$$\begin{aligned}y &= \lim_{n \doteq \infty} \left[n \left\{ 1 + \frac{1}{n}x + \frac{1}{n} \left(\frac{1}{n} - 1 \right) \frac{x^2}{\underline{2}} + \dots \right\} - n \right] \\ &= \lim_{n \doteq \infty} \left[x + \left(\frac{1}{n} - 1 \right) \frac{x^2}{\underline{2}} + \left(\frac{1}{n} - 1 \right) \left(\frac{1}{n} - 2 \right) \frac{x^3}{\underline{3}} + \dots \right] \\ &= x - \frac{x^2}{\underline{2}} + \frac{\underline{2}x^3}{\underline{3}} - \frac{\underline{3}x^4}{\underline{4}} + \dots\end{aligned}$$

$$\therefore \log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

This series is known as the *logarithmic series*. It is convergent only if x lies between -1 and $+1$, or is equal to $+1$. Even within these limits it converges rather slowly, and for these reasons it is not well adapted to the computation of logarithms. A more convenient series is obtained as follows.

Calculation of Logarithms. The equation

$$\log_e(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \dots \quad (1)$$

holds true for all values of y numerically less than 1 ; therefore, if it holds true for any particular value of y less than 1 , it will hold true when we put $-y$ for y ; this gives

$$\log_e(1-y) = -y - \frac{y^2}{2} - \frac{y^3}{3} - \frac{y^4}{4} - \dots \quad (2)$$

Subtracting (2) from (1), since

$$\log_e(1+y) - \log_e(1-y) = \log_e\left(\frac{1+y}{1-y}\right),$$

we find
$$\log_e\left(\frac{1+y}{1-y}\right) = 2\left(y + \frac{y^3}{3} + \frac{y^5}{5} + \dots\right).$$

Put
$$y = \frac{1}{2z+1};$$

then
$$\frac{1+y}{1-y} = \frac{z+1}{z},$$

and
$$\begin{aligned} \log_e\left(\frac{z+1}{z}\right) &= \log_e(z+1) - \log_e z \\ &= 2\left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \dots\right). \end{aligned}$$

This series is convergent for all positive values of z .

Logarithms to any base a can be calculated by the series :

$$\log_a(z+1) - \log_a z = \frac{2}{\log_a a} \left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \dots \right).$$

EXAMPLE. Calculate $\log_e 2$ to five places of decimals.

Let $z = 1$; then $z + 1 = 2$, $2z + 1 = 3$,

and $\log_e 2 = \frac{2}{3} + \frac{2}{3 \times 3^3} + \frac{2}{5 \times 3^5} + \frac{2}{7 \times 3^7} + \dots$

The work may be arranged as follows :

$$\begin{array}{r} 3 \overline{) 2.000000} \\ 9 \overline{) 0.666667} \div 1 = 0.666667 \\ 9 \overline{) 0.074074} \div 3 = 0.024691 \\ 9 \overline{) 0.008230} \div 5 = 0.001646 \\ 9 \overline{) 0.000914} \div 7 = 0.000131 \\ 9 \overline{) 0.000102} \div 9 = 0.000011 \\ 0.000011 \div 11 = 0.000001 \\ \log_e 2 = 0.693147 \end{array}$$

NOTE. In calculating logarithms the accuracy of the work may be tested every time we come to a composite number by adding the logarithms of the several factors. In fact, the logarithms of composite numbers are best found by addition, and then only the logarithms of prime numbers need be computed by the series.

EXERCISE XXVI

1. Calculate to five places of decimals $\log_e 3$.
2. Calculate to five places of decimals $\log_e 5$.
3. Calculate to five places of decimals $\log_e 7$.
4. Calculate to ten places of decimals $\log_{10} 10$.
5. Calculate to five places of decimals $\log_{10} 2$, $\log_{10} e$, $\log_{10} 11$.

SECTION XLV

TRIGONOMETRIC FUNCTIONS OF SMALL ANGLES

Let $\angle AOP$ (Fig. 74) be any angle less than 90° and x its circular measure. Describe a circle of unit radius about O as a centre and take $\angle AOP' = -\angle AOP$. Draw the tangents to the circle at P and P' , meeting OA in T . Then, from Geometry,

$$\begin{aligned} \text{chord } PP' &< \text{arc } PP' \\ &< PT + P'T, \end{aligned}$$

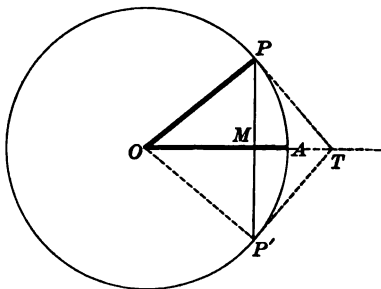


FIG. 74

or, by dividing by 2,

$$MP < \text{arc } AP < PT,$$

or $\sin x < x < \tan x$.

Hence, dividing by $\sin x$,

$$\begin{aligned} 1 &< \frac{x}{\sin x} < \sec x, \\ 1 &> \frac{\sin x}{x} > \cos x. \end{aligned} \tag{1}$$

Then $\frac{\sin x}{x}$ lies between $\cos x$ and 1.

If now the angle x is constantly diminished, $\cos x$ approaches the value 1.

Accordingly, the limit of $\frac{\sin x}{x}$, as x approaches 0, is 1. In other words, if x is a very small angle, then $\frac{\sin x}{x}$ differs from 1 by a small value ϵ ; and this small value ϵ approaches 0 as x approaches 0.

EXAMPLE. To find the sine and cosine of $1'$.

If x is the circular measure of $1'$,

$$x = \frac{2\pi}{360 \times 60} = \frac{3.14159+}{10800} = 0.00029088+,$$

the next figure in x being 8.

Now $\sin x > 0$ but $< x$; hence, $\sin 1'$ lies between 0 and 0.000290889.

Again, $\cos 1' = \sqrt{1 - \sin^2 1'} > \sqrt{1 - (0.0003)^2} > 0.9999999$.

Hence, $\cos 1' = 0.9999999+.$

But, from (1), $\sin x > x \cos x$.

$$\begin{aligned} \therefore \sin 1' &> 0.000290888 \times 0.9999999 \\ &> 0.000290888 (1 - 0.0000001) \\ &> 0.000290888 - 0.0000000000290888 \\ &> 0.000290887. \end{aligned}$$

Hence, $\sin 1'$ lies between 0.000290887 and 0.000290889; that is, to eight places of decimals

$$\sin 1' = 0.00029088+,$$

the next figure being 7 or 8.

EXERCISE XXVII

Given $\pi = 3.141592653589$:

1. Compute $\sin 1'$, $\cos 1'$, and $\tan 1'$ to eleven places of decimals.
2. Compute $\sin 2'$ by the same method, and also by the formula $\sin 2x = 2 \sin x \cos x$. Carry the operations to nine places of decimals. Do the two results agree?
3. Compute $\sin 1^\circ$ to four places of decimals.

4. From the formula $\cos x = 1 - 2 \sin^2 \frac{x}{2}$, show that

$$\cos x > 1 - \frac{x^2}{2}.$$

5. Show by aid of a table of natural sines that $\sin x$ and x agree to four places of decimals for all angles less than $4^\circ 40'$.

6. If the values of $\log x$ and $\log \sin x$ agree to five decimal places, find from a table the greatest value x can have.

SECTION XLVI

SIMPSON'S METHOD OF CONSTRUCTING A TRIGONOMETRIC TABLE

By Sect. XXXII, p. 59,

$$\sin(A + B) + \sin(A - B) = 2 \sin A \cos B.$$

If we put $A = x + 2y$, $B = y$,

we have $\sin(x + 3y) + \sin(x + y) = 2 \sin(x + 2y) \cos y$,

or $\sin(x + 3y) = 2 \sin(x + 2y) \cos y - \sin(x + y)$.

Similarly, $\cos(x + 3y) = 2 \cos(x + 2y) \cos y - \cos(x + y)$. (1)

If $y = 1'$, the last two equations become

$$\sin(x + 3') = 2 \sin(x + 2') \cos 1' - \sin(x + 1'),$$

$$\cos(x + 3') = 2 \cos(x + 2') \cos 1' - \cos(x + 1').$$

Hence, taking x successively equal to $-1'$, $0'$, $1'$, $2'$, $\dots$, we obtain

$$\sin 2' = 2 \sin 1' \cos 1',$$

$$\sin 3' = 2 \sin 2' \cos 1' - \sin 1',$$

$$\sin 4' = 2 \sin 3' \cos 1' - \sin 2',$$

$$\vdots$$

$$\cos 2' = 2 \cos^2 1' - 1,$$

$$\cos 3' = 2 \cos 2' \cos 1' - \cos 1',$$

$$\cos 4' = 2 \cos 3' \cos 1' - \cos 2',$$

$$\vdots$$

Since $\sin 1'$ and $\cos 1'$ are known, these equations enable us to compute step by step the sine and cosine of any angle. The tangent may then be found in each case as the quotient of the sine divided by the cosine.

This process need be carried only as far as 30° . For

$$\sin(30^\circ + x) + \sin(30^\circ - x) = 2 \sin 30^\circ \cos x = \cos x,$$

$$\cos(30^\circ + x) - \cos(30^\circ - x) = -2 \sin 30^\circ \sin x = -\sin x.$$

$$\therefore \sin(30^\circ + x) = \cos x - \sin(30^\circ - x),$$

$$\cos(30^\circ + x) = -\sin x + \cos(30^\circ - x).$$

Moreover, the sines and cosines need be calculated only to 45° , since

$$\sin(45^\circ + x) = \cos(45^\circ - x),$$

$$\cos(45^\circ + x) = \sin(45^\circ - x).$$

In using this method, the multiplication by $\cos 1'$, which occurs at each step, can be simplified by noting that

$$\cos 1' = 0.9999999 = 1 - 0.0000001.$$

NOTE. Simpson's method is superseded in actual practice by much more rapid and convenient processes in which we employ the expansions of the trigonometric functions in infinite series.

EXERCISE XXVIII

1. Compute the sine and cosine of $6'$ to seven decimal places.

In Formula (1) let $y = 1^\circ$. Assuming

$$\sin 1^\circ = 0.017454+, \quad \cos 1^\circ = 0.999848+;$$

2. Compute the sine and cosine of two degrees.
3. Compute the sine and cosine of three degrees.
4. Compute the sine and cosine of four degrees.
5. Compute the sine and cosine of five degrees.

SECTION XLVII

DE MOIVRE'S THEOREM

Expressions of the form

$$\cos x + i \sin x,$$

when $i = \sqrt{-1}$, play an important part in modern analysis.

Given two such expressions,

$$\cos x + i \sin x, \quad \cos y + i \sin y$$

their product is

$$\begin{aligned} & (\cos x + i \sin x)(\cos y + i \sin y) \\ &= \cos x \cos y - \sin x \sin y + i(\cos x \sin y + \sin x \cos y) \\ &= \cos(x + y) + i \sin(x + y). \end{aligned}$$

Hence, the product of two expressions of the form

$$\cos x + i \sin x, \quad \cos y + i \sin y$$

is an expression of the same form in which x or y is replaced by $x + y$. In other words, the angle which enters into such a product is the sum of the angles of the factors.

If x and y are equal, we have at once, from the preceding,

$$(\cos x + i \sin x)^2 = \cos 2x + i \sin 2x;$$

and again,

$$\begin{aligned} (\cos x + i \sin x)^3 &= (\cos x + i \sin x)^2 (\cos x + i \sin x) \\ &= (\cos 2x + i \sin 2x)(\cos x + i \sin x) \\ &= \cos 3x + i \sin 3x. \end{aligned}$$

Similarly,

$$(\cos x + i \sin x)^4 = \cos 4x + i \sin 4x,$$

and in general, if n is a positive integer,

$$(\cos x + i \sin x)^n = \cos nx + i \sin nx. \quad (1)$$

Hence,

To raise the expression $\cos x + i \sin x$ to the n th power when n is a positive integer, we have only to multiply the angle x by n .

Again, if n is a positive integer as before,

$$\left(\cos \frac{x}{n} + i \sin \frac{x}{n} \right)^n = \cos x + i \sin x.$$

$$\therefore (\cos x + i \sin x)^{\frac{1}{n}} = \cos \frac{x}{n} + i \sin \frac{x}{n}.$$

Since, however, x may be increased by any integral multiple of 2π without changing $\cos x + i \sin x$, it follows that all the n expressions,

$$\cos \frac{x}{n} + i \sin \frac{x}{n}, \quad \cos \frac{x + 2\pi}{n} + i \sin \frac{x + 2\pi}{n},$$

$$\cos \frac{x + 4\pi}{n} + i \sin \frac{x + 4\pi}{n}, \dots,$$

$$\cos \frac{x + (n-1)2\pi}{n} + i \sin \frac{x + (n-1)2\pi}{n},$$

are n th roots of $\cos x + i \sin x$. There are no other roots, since

$$\cos \frac{x + n2\pi}{n} + i \sin \frac{x + n2\pi}{n}$$

$$= \cos \left(\frac{x}{n} + 2\pi \right) + i \sin \left(\frac{x}{n} + 2\pi \right) = \cos \frac{x}{n} + i \sin \frac{x}{n},$$

$$\text{and } \cos \frac{x + (n+1)2\pi}{n} + i \sin \frac{x + (n+1)2\pi}{n}$$

$$= \cos \left(\frac{x + 2\pi}{n} + 2\pi \right) + i \sin \left(\frac{x + 2\pi}{n} + 2\pi \right)$$

$$= \cos \frac{x + 2\pi}{n} + i \sin \frac{x + 2\pi}{n},$$

and so on.

Hence, if n is a positive integer,

$$\begin{aligned}
 & (\cos x + i \sin x)^{\frac{1}{n}} \\
 &= \cos \frac{x + 2k\pi}{n} + i \sin \frac{x + 2k\pi}{n} \quad (k = 0, 1, 2, \dots, n-1). \quad (2)
 \end{aligned}$$

From (1) and (2) it follows at once that if m and n are positive integers,

$$\begin{aligned}
 (\cos x + i \sin x)^{\frac{m}{n}} &= \{(\cos x + i \sin x)^{\frac{1}{n}}\}^m \\
 &= \cos \frac{m}{n}(x + 2k\pi) + i \sin \frac{m}{n}(x + 2k\pi) \\
 &\quad (k = 0, 1, 2, \dots, n-1). \quad (3)
 \end{aligned}$$

Finally, if $-\frac{m}{n}$ is a negative fraction,

$$(\cos x + i \sin x)^{-\frac{m}{n}} = \frac{1}{(\cos x + i \sin x)^{\frac{m}{n}}}.$$

$$\begin{aligned}
 \text{But } \frac{1}{\cos x + i \sin x} &= \frac{\cos x - i \sin x}{(\cos x + i \sin x)(\cos x - i \sin x)} \\
 &= \frac{\cos x - i \sin x}{\cos^2 x + \sin^2 x} \\
 &= \cos x - i \sin x \\
 &= \cos(-x) + i \sin(-x).
 \end{aligned}$$

$$\begin{aligned}
 \text{Hence, } (\cos x + i \sin x)^{-\frac{m}{n}} &= \{\cos(-x) + i \sin(-x)\}^{\frac{m}{n}} \\
 &= \cos \frac{m}{n}(-x + 2k\pi) + i \sin \frac{m}{n}(-x + 2k\pi), \\
 &\quad (k = 0, 1, 2, \dots, n-1) \\
 &= \cos \left\{ -\frac{m}{n}(x + 2k\pi) \right\} + i \sin \left\{ -\frac{m}{n}(x + 2k\pi) \right\}, \\
 &\quad (k = 0, 1, 2, \dots, n-1). \quad (4)
 \end{aligned}$$

Consequently, if n is a positive or negative integer or fraction,

$$(\cos x + i \sin x)^n = \cos [n(x + 2k\pi)] + i \sin [n(x + 2k\pi)],$$

$$(k = 0, 1, 2, \dots, n-1). \quad (5)$$

EXAMPLE. Find the three cube roots of -1 .

We have $-1 = \cos 180^\circ + i \sin 180^\circ$.

$$\therefore (-1)^{\frac{1}{3}} = \cos \frac{180^\circ + 2k\pi}{3} + i \sin \frac{180^\circ + 2k\pi}{3} \quad (k = 0, 1, 2).$$

For the three cube roots of -1 we find, therefore,

$$\cos 60^\circ + i \sin 60^\circ, \quad \cos 180^\circ + i \sin 180^\circ, \quad \cos 300^\circ + i \sin 300^\circ,$$

or $\frac{1 + i\sqrt{3}}{2}, \quad -1, \quad \frac{1 - i\sqrt{3}}{2}.$

By aid of De Moivre's Theorem, we may express $\sin n\theta$ and $\cos n\theta$, when n is an integer, in terms of $\sin \theta$ and $\cos \theta$.

$$\begin{aligned} \text{Thus,} \quad \cos n\theta + i \sin n\theta &= (\cos \theta + i \sin \theta)^n \\ &= \cos^n \theta + in \cos^{n-1} \theta \sin \theta + i^2 \frac{n(n-1)}{2} \cos^{n-2} \theta \sin^2 \theta \\ &\quad + i^3 \frac{n(n-1)(n-2)}{3} \cos^{n-3} \theta \sin^3 \theta + \dots \end{aligned}$$

Or, since $i^2 = -1$, $i^3 = -i$, $i^4 = +1$, $\dots$,

$$\begin{aligned} \cos n\theta + i \sin n\theta &= \cos^n \theta + in \cos^{n-1} \theta \sin \theta \\ &\quad - \frac{n(n-1)}{2} \cos^{n-2} \theta \sin^2 \theta - i \frac{n(n-1)(n-2)}{3} \cos^{n-3} \theta \sin^3 \theta + \dots \end{aligned}$$

Equating now the real parts and the imaginary parts separately, we obtain

$$\begin{aligned} \cos n\theta &= \cos^n \theta - \frac{n(n-1)}{2} \cos^{n-2} \theta \sin^2 \theta \\ &\quad + \frac{n(n-1)(n-2)(n-3)}{4} \cos^{n-4} \theta \sin^4 \theta - \dots, \end{aligned}$$

$$\sin n\theta = n \cos^{n-1}\theta \sin \theta - \frac{n(n-1)(n-2)}{3} \cos^{n-3}\theta \sin^3\theta \\ + \frac{n(n-1)(n-2)(n-3)(n-4)}{5} \cos^{n-5}\theta \sin^5\theta - \dots$$

EXERCISE XXIX

1. Find the six 6th roots of -1 ; of $+1$.
2. Find the three cube roots of i .
3. Find the four 4th roots of $-i$.
4. Express $\sin 4\theta$ and $\cos 4\theta$ in terms of $\sin \theta$ and $\cos \theta$.

SECTION XLVIII**EXPANSION OF $\sin x$, $\cos x$, AND $\tan x$ IN INFINITE SERIES**

Let one radian be denoted simply by 1, and let

$$\cos 1 + i \sin 1 = k.$$

Then $\cos x + i \sin x = (\cos 1 + i \sin 1)^x = k^x$,

and, putting $-x$ for x ,

$$\cos(-x) + i \sin(-x) = \cos x - i \sin x = k^{-x}.$$

That is, $\cos x + i \sin x = k^x$,

and $\cos x - i \sin x = k^{-x}$.

By taking the sum and difference of these two equations, and dividing the sum by 2 and the difference by $2i$, we have

$$\cos x = \frac{1}{2}(k^x + k^{-x}), \quad \sin x = \frac{1}{2i}(k^x - k^{-x}).$$

But $k^x = (e^{\log k})^x = e^{x \log k}$, $k^{-x} = e^{-x \log k}$,

and $e^{x \log k} = 1 + x \log k + \frac{x^2(\log k)^2}{2} + \frac{x^3(\log k)^3}{3} + \dots$,

$$e^{-x \log k} = 1 - x \log k + \frac{x^2 (\log k)^2}{2} - \frac{x^3 (\log k)^3}{3} + \dots$$

$$\therefore \cos x = \frac{1}{2} (k^x + k^{-x}) = 1 + \frac{x^2 (\log k)^2}{2} + \frac{x^4 (\log k)^4}{4} + \dots,$$

$$\sin x = \frac{1}{i} \left\{ x \log k + \frac{x^3 (\log k)^3}{3} + \frac{x^5 (\log k)^5}{5} + \dots \right\}.$$

It only remains to find the value of k , and this can be obtained by dividing the last equation through by x and letting x approach 0 indefinitely.

Then we have

$$\lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right) = \frac{1}{i} \log k.$$

But $\lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right) = 1.$

$$\therefore \log k = i.$$

$$\therefore k = e^i.$$

Therefore, we have

$$\cos x = \frac{1}{2} (e^{xi} + e^{-xi}) = 1 - \frac{x^2}{2} + \frac{x^4}{4} - \frac{x^6}{6} + \dots,$$

$$\sin x = \frac{1}{2i} (e^{xi} - e^{-xi}) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

From the last two series we obtain, by division,

$$\tan x = \frac{\sin x}{\cos x} = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \dots$$

By the aid of these series the trigonometric functions of any angle are readily calculated.

In the computation it must be remembered that x is the *circular measure* of the given angle.

EXERCISE XXX

Verify by the series just obtained that :

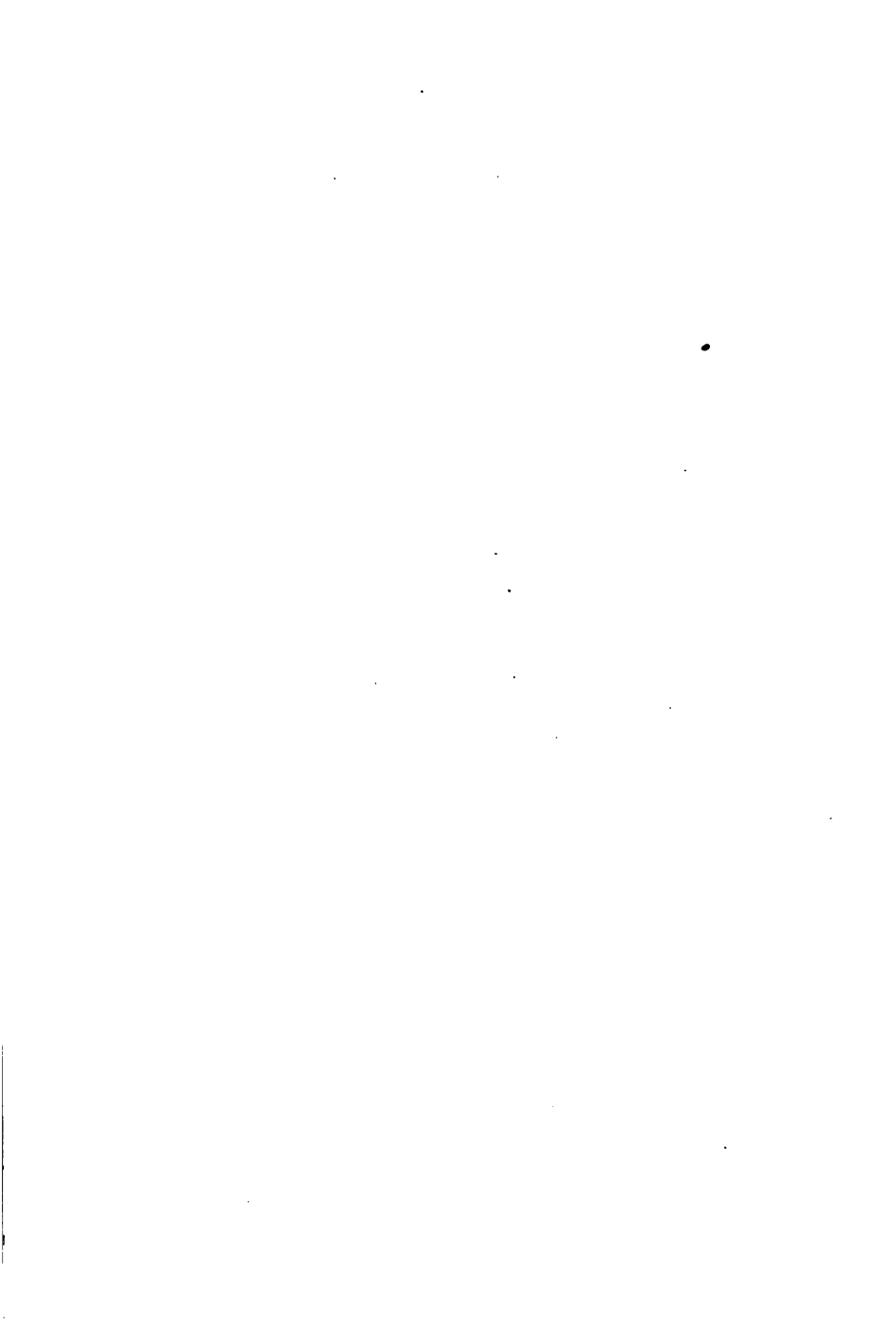
1. $\sin^2 x + \cos^2 x = 1$.
2. $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$.
3. $\sin 2x = 2 \sin x \cos x$.
4. $\cos 2x = 1 - 2 \sin^2 x$.

5. Find the series for $\sec x$ as far as the term containing the 6th power of x .

6. Find the series for $x \cot x$, noting that

$$x \cot x = \frac{x}{\sin x} \cos x.$$

7. Calculate $\sin 10^\circ$ and $\cos 10^\circ$ to five places of decimals.
8. Calculate $\tan 15^\circ$ to five places of decimals.
9. From the exponential value of $\cos x$ show that
$$\cos 3x = 4 \cos^3 x - 3 \cos x.$$
10. From the exponential value of $\sin x$ show that
$$\sin 3x = 3 \sin x - 4 \sin^3 x.$$



FORMULAS

PLANE TRIGONOMETRY

$$1. \sin^2 A + \cos^2 A = 1.$$

$$2. \tan A = \frac{\sin A}{\cos A}.$$

$$3. \begin{cases} \sin A \times \csc A = 1. \\ \cos A \times \sec A = 1. \\ \tan A \times \cot A = 1. \end{cases}$$

$$4. \sin(x + y) = \sin x \cos y + \cos x \sin y.$$

$$5. \cos(x + y) = \cos x \cos y - \sin x \sin y.$$

$$6. \tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}.$$

$$7. \cot(x + y) = \frac{\cot x \cot y - 1}{\cot y + \cot x}.$$

$$8. \sin(x - y) = \sin x \cos y - \cos x \sin y.$$

$$9. \cos(x - y) = \cos x \cos y + \sin x \sin y.$$

$$10. \tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}.$$

$$11. \cot(x - y) = \frac{\cot x \cot y + 1}{\cot y - \cot x}.$$

$$12. \sin 2x = 2 \sin x \cos x.$$

$$13. \cos 2x = \cos^2 x - \sin^2 x.$$

$$14. \tan 2x = \frac{2 \tan x}{1 - \tan^2 x}.$$

$$15. \cot 2x = \frac{\cot^2 x - 1}{2 \cot x}.$$

$$16. \sin \frac{1}{2} z = \pm \sqrt{\frac{1 - \cos z}{2}}.$$

$$17. \cos \frac{1}{2} z = \pm \sqrt{\frac{1 + \cos z}{2}}.$$

$$18. \tan \frac{1}{2} z = \pm \sqrt{\frac{1 - \cos z}{1 + \cos z}}.$$

$$19. \cot \frac{1}{2} z = \pm \sqrt{\frac{1 + \cos z}{1 - \cos z}}.$$

$$20. \sin A + \sin B = 2 \sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B).$$

$$21. \sin A - \sin B = 2 \cos \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B).$$

$$22. \cos A + \cos B = 2 \cos \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B).$$

$$23. \cos A - \cos B = -2 \sin \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B).$$

$$24. \frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan \frac{1}{2}(A + B)}{\tan \frac{1}{2}(A - B)}.$$

$$25. \frac{a}{b} = \frac{\sin A}{\sin B}.$$

$$26. a^2 = b^2 + c^2 - 2bc \cos A.$$

$$27. \frac{a - b}{a + b} = \frac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)}.$$

$$28. \sin \frac{1}{2} A = \sqrt{\frac{(s - b)(s - c)}{bc}}.$$

$$29. \cos \frac{1}{2} A = \sqrt{\frac{s(s - a)}{bc}}.$$

$$30. \tan \frac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

$$31. \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} = r.$$

$$32. \tan \frac{1}{2} A = \frac{r}{s-a}.$$

$$33. F = \frac{1}{2} ac \sin B.$$

$$34. F = \frac{a^2 \sin B \sin C}{2 \sin (B+C)}.$$

$$35. F = \sqrt{s(s-a)(s-b)(s-c)}.$$

$$36. F = \frac{abc}{4R}.$$

$$37. F = \frac{1}{2} r(a+b+c) = rs.$$

ANSWERS

PLANE TRIGONOMETRY

Exercise I. Page 2

- $\frac{1}{3}\pi$; $\frac{1}{4}\pi$; $\frac{5}{8}\pi$; $\frac{11}{12}\pi$; $\frac{1}{16}\pi$; $\frac{1}{18}\pi$; $\frac{5}{24}\pi$.
- 120° ; 135° ; $112^\circ 30'$; $168^\circ 45'$; 84° .
- 0.017453; 0.0002909.
8. 69.166 miles.
- 206,265".
- 57 feet 3.55 inches.
- $\frac{3}{4}\pi$; $\frac{5}{8}\pi$.
10. 3 hours 49 minutes 11 seconds.
- $11^\circ 27' 33''$.
- 9 feet 2 inches.
- $14^\circ 27' 28''$.
- $\frac{7}{165}$ seconds.

Exercise II. Page 5

- $\sin B = \frac{b}{c}$; $\cos B = \frac{a}{c}$; $\tan B = \frac{b}{a}$; $\cot B = \frac{a}{b}$; $\sec B = \frac{c}{a}$; $\csc B = \frac{c}{b}$.
- (i) $\sin = \frac{5}{13}$, $\cos = \frac{12}{13}$, $\tan = \frac{5}{12}$, $\cot = \frac{12}{5}$, $\sec = \frac{13}{12}$, $\csc = \frac{13}{5}$;
 (ii) $\sin = \frac{5}{13}$, $\cos = \frac{12}{13}$, $\tan = \frac{5}{12}$, $\cot = \frac{12}{5}$, $\sec = \frac{13}{12}$, $\csc = \frac{13}{5}$;
 (iii) $\sin = \frac{1}{7}$, $\cos = \frac{12}{13}$, $\tan = \frac{5}{12}$, $\cot = \frac{12}{5}$, $\sec = \frac{13}{12}$, $\csc = \frac{13}{5}$;
 (iv) $\sin = \frac{9}{41}$, $\cos = \frac{40}{41}$, $\tan = \frac{9}{40}$, $\cot = \frac{40}{9}$, $\sec = \frac{41}{40}$, $\csc = \frac{41}{9}$;
 (v) $\sin = \frac{3}{5}$, $\cos = \frac{4}{5}$, $\tan = \frac{3}{4}$, $\cot = \frac{4}{3}$, $\sec = \frac{5}{4}$, $\csc = \frac{5}{3}$;
 (vi) $\sin = \frac{11}{60}$, $\cos = \frac{59}{60}$, $\tan = \frac{11}{59}$, $\cot = \frac{59}{11}$, $\sec = \frac{60}{59}$, $\csc = \frac{60}{11}$.
- The required condition is that $a^2 + b^2 = c^2$. It is.
- (i) $\sin = \frac{2mn}{m^2 + n^2}$, $\cos = \frac{m^2 - n^2}{m^2 + n^2}$, $\tan = \frac{2mn}{m^2 - n^2}$,
 $\cot = \frac{m^2 - n^2}{2mn}$, $\sec = \frac{m^2 + n^2}{m^2 - n^2}$, $\csc = \frac{m^2 + n^2}{2mn}$;
 (ii) $\sin = \frac{2xy}{x^2 + y^2}$, $\cos = \frac{x^2 - y^2}{x^2 + y^2}$, $\tan = \frac{2xy}{x^2 - y^2}$,
 $\cot = \frac{x^2 - y^2}{2xy}$, $\sec = \frac{x^2 + y^2}{x^2 - y^2}$, $\csc = \frac{x^2 + y^2}{2xy}$;

$$(iii) \sin = \frac{q}{s}, \quad \cos = \frac{q}{p}, \quad \tan = \frac{p}{s}, \quad \cot = \frac{s}{p}, \quad \sec = \frac{p}{q}, \quad \csc = \frac{s}{q};$$

$$(iv) \sin = \frac{ms}{qr}, \quad \cos = \frac{mpv}{nqr}, \quad \tan = \frac{ns}{pv},$$

$$\cot = \frac{pv}{ns}, \quad \sec = \frac{nqr}{mpv}, \quad \csc = \frac{qr}{ms}.$$

$$7. \text{ In (iii) } p^2q^2 + q^2s^2 = p^2s^2; \text{ in (iv) } m^2n^2s^2 + m^2p^2v^2 = n^2q^2r^2.$$

$$8. \sin A = \frac{2A}{145} = \cos B; \quad \cos A = \frac{14}{14} = \sin B; \quad \tan A = \frac{2A}{145} = \cot B;$$

$$\cot A = \frac{14}{14} = \tan B; \quad \sec A = \frac{14}{14} = \csc B; \quad \csc A = \frac{14}{14} = \sec B.$$

$$9. \sin A = \frac{2}{14} = \cos B; \quad \cos A = \frac{2}{14} = \sin B; \quad \tan A = \frac{2}{14} = \cot B;$$

$$\cot A = \frac{2}{14} = \tan B; \quad \sec A = \frac{2}{14} = \csc B; \quad \csc A = \frac{2}{14} = \sec B.$$

$$10. \sin A = \frac{1}{14} = \cos B; \quad \cos A = \frac{1}{14} = \sin B; \quad \tan A = \frac{1}{14} = \cot B;$$

$$\cot A = \frac{1}{14} = \tan B; \quad \sec A = \frac{1}{14} = \csc B; \quad \csc A = \frac{1}{14} = \sec B.$$

$$11. \sin A = \frac{\sqrt{p^2 + q^2}}{p + q} = \cos B; \quad \cos A = \frac{\sqrt{2pq}}{p + q} = \sin B;$$

$$\tan A = \frac{\sqrt{p^2 + q^2}}{\sqrt{2pq}} = \cot B; \quad \cot A = \frac{\sqrt{2pq}}{\sqrt{p^2 + q^2}} = \tan B;$$

$$\sec A = \frac{p + q}{\sqrt{2pq}} = \csc B; \quad \csc A = \frac{p + q}{\sqrt{p^2 + q^2}} = \sec B.$$

$$12. \sin A = \frac{\sqrt{p^2 + pq}}{p + q} = \cos B; \quad \cos A = \frac{\sqrt{q^2 + pq}}{p + q} = \sin B;$$

$$\tan A = \sqrt{\frac{p}{q}} = \cot B; \quad \cot A = \sqrt{\frac{q}{p}} = \tan B;$$

$$\sec A = \frac{p + q}{\sqrt{q^2 + pq}} = \csc B; \quad \csc A = \frac{p + q}{\sqrt{p^2 + pq}} = \sec B.$$

$$13. \sin A = \frac{p - q}{p + q} = \cos B; \quad \cos A = \frac{2\sqrt{pq}}{p + q} = \sin B;$$

$$\tan A = \frac{p - q}{2\sqrt{pq}} = \cot B; \quad \cot A = \frac{2\sqrt{pq}}{p - q} = \tan B;$$

$$\sec A = \frac{p + q}{2\sqrt{pq}} = \csc B; \quad \csc A = \frac{p + q}{p - q} = \sec B.$$

$$14. \sin A = \frac{1}{2}\sqrt{5}; \quad \cos A = \frac{1}{2}\sqrt{5}; \quad \tan A = 2; \quad \cot A = \frac{1}{2};$$

$$\sec A = \sqrt{5}; \quad \csc A = \frac{1}{2}\sqrt{5}.$$

$$15. \sin A = \frac{1}{2}; \quad \cos A = \frac{1}{2}\sqrt{5}; \quad \tan A = \frac{1}{2}\sqrt{5}; \quad \cot A = \frac{1}{2}\sqrt{5};$$

$$\sec A = \frac{1}{2}\sqrt{5}; \quad \csc A = \frac{1}{2}.$$

16. $\sin A = \frac{1}{3}(5 + \sqrt{7})$; $\cos A = \frac{1}{3}(5 - \sqrt{7})$;
 $\tan A = \frac{1}{3}(16 + 5\sqrt{7})$; $\cot A = \frac{1}{3}(16 - 5\sqrt{7})$;
 $\sec A = \frac{1}{3}(5 + \sqrt{7})$; $\csc A = \frac{1}{3}(5 - \sqrt{7})$.
17. $\sin A = \frac{1}{15}(\sqrt{31} + 1)$; $\cos A = \frac{1}{15}(\sqrt{31} - 1)$;
 $\tan A = \frac{1}{15}(16 + \sqrt{31})$; $\cot A = \frac{1}{15}(16 - \sqrt{31})$;
 $\sec A = \frac{1}{15}(\sqrt{31} + 1)$; $\csc A = \frac{1}{15}(\sqrt{31} - 1)$.
18. $a = 12.3$. 20. $a = 9$. 22. $c = 40$.
 19. $b = 1.54$. 21. $b = 68$. 23. $c = 229.62$.
24. Construct a rt. Δ with legs equal to 3 and 2, respectively; then construct a similar Δ with hypotenuse equal to 6.
28. $a = 1.5$ miles; $b = 2$ miles.
30. $a = 0.342$, $b = 0.940$; $a = 1.368$, $b = 3.760$. 31. 142.926 yards.

Exercise III. Page 9

5. Through A (Fig. 3) draw a tangent, and take AT equal to 3; the angle AOT is the required angle.
6. From O (Fig. 3) as a centre, with a radius equal to 2, describe an arc cutting at S the tangent drawn through B ; the angle AOS is the required angle.
7. In Fig. 3, take OM equal to $\frac{1}{2}$, and erect $MP \perp OA$, intersecting the circumference at P ; the angle POM is the required angle.
8. Since $\sin x = \cos z$, $OM = PM$ (Fig. 3), and $x = 45^\circ$; hence, construct x equal to 45° .
9. Construct a rt. Δ with one leg equal to twice the other; the angle opposite the longer leg is the required angle.
10. Divide OA (Fig. 3) into four equal parts; at the first point of division from O erect a perpendicular meeting the circumference at some point P . Draw OP ; the angle AOP is the required angle.
12. $x = 18^\circ$. 21. $r \sin x$. 22. $a = mc$; $b = nc$.

Exercise IV. Page 12

1. $\cos 60^\circ$; $\sin 45^\circ$; $\cot 1^\circ$; $\tan 75^\circ$;
 $\sec 71^\circ 50'$; $\sin 52^\circ 38'$; $\tan 7^\circ 41'$; $\sec 35^\circ 14'$.
2. $\cos 30^\circ$; $\sin 15^\circ$; $\cot 33^\circ$; $\tan 6^\circ$;
 $\sec 20^\circ 58'$; $\sin 4^\circ 21'$; $\tan 0^\circ 1'$; $\sec 44^\circ 59'$.

- | | | | |
|----------------------------|-----------------|---------------------|------------------------------|
| 3. $\frac{1}{2}\sqrt{3}$. | 6. 30° . | 9. $22^\circ 30'$. | 12. $\frac{90^\circ}{n+1}$. |
| 4. 45° . | 7. 90° . | 10. 18° . | |
| 5. 30° . | 8. 60° . | 11. 10° . | |

Exercise VI. Page 16

1. $\cos A = \frac{5}{13}$; $\tan A = \frac{12}{5}$; $\cot A = \frac{5}{12}$; $\sec A = \frac{13}{5}$; $\csc A = \frac{13}{12}$.
2. $\cos A = 0.6$; $\tan A = 1.3333$; $\cot A = 0.75$; $\sec A = 1.6667$; $\csc A = 1.25$.
3. $\sin A = \frac{1}{2}$; $\tan A = \frac{1}{\sqrt{3}}$; $\cot A = \sqrt{3}$; $\sec A = \frac{2}{\sqrt{3}}$; $\csc A = \frac{2}{1}$.
4. $\sin A = 0.96$; $\tan A = 3.4286$; $\cot A = 0.2917$; $\sec A = 3.5714$; $\csc A = 1.0417$.
5. $\sin A = 0.8$; $\cos A = 0.6$; $\cot A = 0.75$; $\sec A = 1.6667$; $\csc A = 1.25$.
6. $\sin A = \frac{1}{2}\sqrt{2}$; $\cos A = \frac{1}{2}\sqrt{2}$; $\tan A = 1$; $\sec A = \sqrt{2}$; $\csc A = \sqrt{2}$.
7. $\sin A = 0.90$; $\cos A = 0.45$; $\tan A = 2$; $\sec A = 2.22$; $\csc A = 1.11$.
8. $\sin A = \frac{1}{2}\sqrt{3}$; $\cos A = \frac{1}{2}$; $\tan A = \sqrt{3}$; $\cot A = \frac{1}{\sqrt{3}}$; $\csc A = \frac{2}{\sqrt{3}}$.
9. $\sin A = \frac{1}{2}\sqrt{2}$; $\cos A = \frac{1}{2}\sqrt{2}$; $\tan A = 1$; $\cot A = 1$; $\sec A = \sqrt{2}$.
10. $\cos A = \sqrt{1-m^2}$; $\tan A = \frac{m}{\sqrt{1-m^2}}$; $\cot A = \frac{1}{m}\sqrt{1-m^2}$; $\sec A = \frac{1}{\sqrt{1-m^2}}$; $\csc A = \frac{1}{m}$.
11. $\cos A = \frac{1-m^2}{1+m^2}$; $\tan A = \frac{2m}{1-m^2}$; $\cot A = \frac{1-m^2}{2m}$; $\sec A = \frac{1+m^2}{1-m^2}$; $\csc A = \frac{1+m^2}{2m}$.
12. $\sin A = \frac{m^2-n^2}{m^2+n^2}$; $\tan A = \frac{m^2-n^2}{2mn}$; $\cot A = \frac{2mn}{m^2-n^2}$; $\sec A = \frac{m^2+n^2}{2mn}$; $\csc A = \frac{m^2+n^2}{m^2-n^2}$.
13. $\sin = \frac{1}{2}\sqrt{2}$; $\cos = \frac{1}{2}\sqrt{2}$; $\cot = 1$; $\sec = \sqrt{2}$; $\csc = \sqrt{2}$.
14. $\cos = \frac{1}{2}\sqrt{3}$; $\tan = \frac{1}{\sqrt{3}}$; $\cot = \sqrt{3}$; $\sec = \frac{2}{\sqrt{3}}$; $\csc = 2$.
15. $\sin = \frac{1}{2}\sqrt{3}$; $\cos = \frac{1}{2}$; $\tan = \sqrt{3}$; $\cot = \frac{1}{\sqrt{3}}$; $\sec = 2$.
16. $\sin = \frac{1}{2}\sqrt{2-\sqrt{3}}$; $\cos = \frac{1}{2}\sqrt{2+\sqrt{3}}$; $\cot = 2+\sqrt{3}$; $\sec = 2(2-\sqrt{3})\sqrt{2+\sqrt{3}}$; $\csc = 2(2+\sqrt{3})\sqrt{2-\sqrt{3}}$.

17. $\sin = \frac{1}{2} \sqrt{2 - \sqrt{2}}$; $\cos = \frac{1}{2} \sqrt{2 + \sqrt{2}}$; $\tan = \sqrt{2} - 1$;
 $\sec = (2 - \sqrt{2}) \sqrt{2 + \sqrt{2}}$; $\csc = (2 + \sqrt{2}) \sqrt{2 - \sqrt{2}}$.
18. $\cos = 1$; $\tan = 0$; $\cot = \infty$; $\sec = 1$; $\csc = \infty$.
19. $\cos = 0$; $\tan = \infty$; $\cot = 0$; $\sec = \infty$; $\csc = 1$.
20. $\sin = 1$; $\cos = 0$; $\cot = 0$; $\sec = \infty$; $\csc = 1$.
21. $\cos A = \sqrt{1 - \sin^2 A}$; $\tan A = \frac{\sin A}{\sqrt{1 - \sin^2 A}}$; $\cot A = \frac{\sqrt{1 - \sin^2 A}}{\sin A}$;
 $\sec A = \frac{1}{\sqrt{1 - \sin^2 A}}$; $\csc A = \frac{1}{\sin A}$.
22. $\sin A = \sqrt{1 - \cos^2 A}$; $\tan A = \frac{\sqrt{1 - \cos^2 A}}{\cos A}$; $\cot A = \frac{\cos A}{\sqrt{1 - \cos^2 A}}$;
 $\sec A = \frac{1}{\cos A}$; $\csc A = \frac{1}{\sqrt{1 - \cos^2 A}}$.
23. $\sin A = \frac{\tan A}{\sqrt{1 + \tan^2 A}}$; $\cos A = \frac{1}{\sqrt{1 + \tan^2 A}}$; $\cot A = \frac{1}{\tan A}$;
 $\sec A = \sqrt{1 + \tan^2 A}$; $\csc A = \frac{\sqrt{1 + \tan^2 A}}{\tan A}$.
24. $\sin A = \frac{1}{\sqrt{1 + \cot^2 A}}$; $\cos A = \frac{\cot A}{\sqrt{1 + \cot^2 A}}$; $\tan A = \frac{1}{\cot A}$;
 $\sec A = \frac{\sqrt{1 + \cot^2 A}}{\cot A}$; $\csc A = \sqrt{1 + \cot^2 A}$.
25. $\sin A = \frac{1}{3} \sqrt{5}$; $\cos A = \frac{2}{3} \sqrt{5}$. 27. $\sin A = \frac{2}{5}$; $\cos A = \frac{4}{5}$.
26. $\sin A = \frac{1}{4} \sqrt{15}$; $\tan A = \sqrt{15}$. 28. $\frac{1 - 3 \cos^2 A + 3 \cos^4 A}{\cos^2 A - \cos^4 A}$.

Exercise VII. Page 18

- | | | | |
|------------------------------------|----------------------|--------------------------------------|----------------------|
| 1. $x = 45^\circ$. | 6. $x = 45^\circ$. | 11. $x = 30^\circ$. | 16. $x = 45^\circ$. |
| 2. $x = 30^\circ$. | 7. $x = 45^\circ$. | 12. $x = 45^\circ$. | 17. $x = 60^\circ$. |
| 3. $x = 0^\circ$, or 60° . | 8. $x = 45^\circ$. | 13. $x = 0^\circ$, or 60° . | |
| 4. $x = 45^\circ$. | 9. $x = 60^\circ$. | 14. $x = 30^\circ$. | |
| 5. $x = 60^\circ$. | 10. $x = 60^\circ$. | 15. $x = 30^\circ$, or 45° . | |

Exercise VIII. Page 24

1. $c = \frac{b}{\cos A}$. 2. $c = \frac{a}{\sin A}$. 3. $b = c \cos A$. 4. $c = \frac{a}{\sin A}$.
5. $A = 90^\circ - B$; $a = c \cos B$; $b = c \sin B$.
6. $A = 90^\circ - B$; $a = b \cot B$; $c = \frac{b}{\sin B}$.
7. $A = 90^\circ - B$; $b = a \tan B$; $c = \frac{a}{\cos B}$.
8. $\cos A = \frac{b}{c}$; $B = 90^\circ - A$; $a = \sqrt{(c+b)(c-b)}$.

Exercise IX. Page 28

31. $c = 7.8112$; $A = 39^\circ 48'$; $B = 50^\circ 12'$; $F = 15$.
32. $b = 69.997$; $A = 30' 12''$; $B = 89^\circ 29' 48''$; $F = 21.525$.
33. $a = 1.1886$; $A = 43^\circ 20'$; $B = 46^\circ 40'$; $F = 0.74876$.
34. $b = 21.249$; $c = 22.372$; $B = 71^\circ 46'$; $F = 74.372$.
35. $a = 6.6882$; $c = 13.738$; $B = 60^\circ 52'$; $F = 40.129$.
36. $a = 63.859$; $b = 23.369$; $B = 20^\circ 6'$; $F = 746.15$.
37. $a = 19.40$; $b = 18.778$; $A = 45^\circ 56'$; $F = 182.15$.
38. $b = 53.719$; $c = 71.377$; $A = 41^\circ 11'$; $F = 1262.4$.
39. $a = 12.981$; $c = 15.796$; $A = 55^\circ 16'$; $F = 58.416$.
40. $a = 0.58046$; $b = 8.442$; $A = 3^\circ 56'$; $F = 2.4501$.
41. $F = \frac{1}{2} c^2 \sin A \cos A$. 43. $F = \frac{1}{2} b^2 \tan A$.
42. $F = \frac{1}{2} a^2 \cot A$. 44. $F = \frac{1}{2} a \sqrt{c^2 - a^2}$.
45. $b = 11.6$; $c = 15.315$; $A = 40^\circ 45' 48''$; $B = 49^\circ 14' 12''$.
46. $a = 7.2$; $c = 8.7658$; $A = 55^\circ 13' 20''$; $B = 34^\circ 46' 40''$.
47. $a = 3.6474$; $b = 6.58$; $c = 7.5233$; $B = 61^\circ$.
48. $a = 10.283$; $b = 19.449$; $A = 27^\circ 52'$; $B = 62^\circ 8'$.
49. $19^\circ 28' 17''$ and $70^\circ 31' 43''$. 52. $36^\circ 52' 12''$ and $53^\circ 7' 48''$.
50. 3 and 5.1961. 53. 212.1 feet.
51. $a = c \cos \frac{90^\circ}{n+1}$; 54. 732.22 feet.
- $b = c \sin \frac{90^\circ}{n+1}$. 55. 3270 feet.
56. 37.3 feet.

57. $1^{\circ} 25' 56''$. 58. $59^{\circ} 44' 35''$. 59. 95.34 feet.
 60. 7.0712 miles in each direction.
 61. 20.88 feet. 63. 685.9 feet. 65. 140 feet.
 62. 56.65 feet. 64. 136.6 feet. 66. 84.74 feet.

Exercise X. Page 33

1. $C = 2(90^{\circ} - A)$; $c = 2a \cos A$; $h = a \sin A$.
 2. $A = \frac{1}{2}(180^{\circ} - C)$; $c = 2a \cos A$; $h = a \sin A$.
 3. $C = 2(90^{\circ} - A)$; $a = \frac{c}{2 \cos A}$; $h = a \sin A$.
 4. $A = \frac{1}{2}(180^{\circ} - C)$; $a = \frac{c}{2 \cos A}$; $h = a \sin A$.
 5. $C = 2(90^{\circ} - A)$; $a = \frac{h}{\sin A}$; $c = 2a \cos A$.
 6. $A = \frac{1}{2}(180^{\circ} - C)$; $a = \frac{h}{\sin A}$; $c = 2a \cos A$.
 7. $\sin A = \frac{h}{a}$; $C = 2(90^{\circ} - A)$; $c = 2a \cos A$.
 8. $\tan A = \frac{2h}{c}$; $C = 2(90^{\circ} - A)$; $a = \frac{h}{\sin A}$.
 9. $A = 67^{\circ} 22' 50''$; $C = 45^{\circ} 14' 20''$; $h = 13.2$.
 10. $c = 0.21943$; $h = 0.27384$; $F = 0.03004$.
 11. $a = 2.055$; $h = 1.6852$; $F = 1.9819$.
 12. $a = 7.706$; $c = 3.6676$; $F = 13.725$.
 13. $A = 79^{\circ} 36' 30''$; $C = 20^{\circ} 47'$; $c = 2.4206$.
 14. $A = 77^{\circ} 19' 11''$; $C = 25^{\circ} 21' 38''$; $a = 20.5$.
 15. $A = 26^{\circ} 27' 47''$; $C = 129^{\circ} 4' 26''$; $a = 81.41$; $h = 35$.
 16. $A = 81^{\circ} 12' 9''$; $C = 17^{\circ} 35' 42''$; $a = 17$; $c = 5.2$.
 17. $F = \frac{1}{4}c \sqrt{4a^2 - c^2}$. 22. 0.76536.
 18. $F = a^2 \sin \frac{1}{2}C \cos \frac{1}{2}C$. 23. $94^{\circ} 20'$.
 19. $F = a^2 \sin A \cos A$. 24. 2.7261.
 20. $F = h^2 \tan \frac{1}{2}C$. 25. $38^{\circ} 56' 33''$.
 21. 28.284 feet; 4525.44 square feet. 26. 37.699.

Exercise XI. Page 35

- | | | |
|---|-----------------|----------------|
| 1. $r = 1.618$; | $h = 1.5388$; | $F = 7.694$. |
| 2. $h = 0.9848$; | $p = 6.2514$; | $F = 3.0782$. |
| 3. $h = 19.754$; | $c = 6.257$; | $F = 1236$. |
| 4. $r = 1.0824$; | $c = 0.82842$; | $F = 3.3137$. |
| 5. $r = 2.5933$; | $h = 2.4882$; | $c = 1.4615$. |
| 6. $r = 1.5994$; | $h = 1.441$; | $p = 9.716$. |
| 7. 0.61803. | 11. 0.2238. | 16. 11.636. |
| 8. 0.64984. | 12. 0.310. | 17. 99.640. |
| 9. 0.51764. | 13. 0.82842. | 18. 1.0235. |
| 10. $b = \frac{c}{2 \cos \frac{90^\circ}{n}}$. | 14. 94.63. | 19. 0.635. |
| | 15. 414.97. | |

Exercise XII. Page 45

- Two angles; one in Quadrant I, one in Quadrant II.
- Four values; two in Quadrant I, two in Quadrant IV.
- x may have two values in the first case, and one value in each of the other cases.
- If $\cos x = -\frac{3}{4}$, x is between 90° and 270° ; if $\cot x = 4$, x is between 0° and 90° or between 180° and 270° ; if $\sec x = 80$, x is between 0° and 90° or between 270° and 360° ; if $\csc x = -3$, x is between 180° and 360° .
- In Quadrant III; in Quadrant II; in Quadrant III.
- 40 angles; 20 positive and 20 negative.
- +, when x is known to be in Quadrant I or IV; -, when x is known to be in Quadrant II or III.
- $\sin x = +\frac{1}{2}\sqrt{2}$; $\tan x = -1$; $\cot x = -1$; $\sec x = -\sqrt{2}$;
 $\csc x = +\sqrt{2}$.
- $\sin x = -\frac{1}{2}\sqrt{3}$; $\cos x = -\frac{1}{2}$; $\cot x = +\frac{1}{2}\sqrt{3}$; $\sec x = -2$;
 $\csc x = -\frac{2}{3}\sqrt{3}$.
- $\sin x = -\frac{4}{5}\sqrt{3}$; $\cos x = \frac{1}{5}$; $\tan x = -4\sqrt{3}$; $\cot x = -\frac{1}{4}\sqrt{3}$;
 $\csc x = -\frac{5}{4}\sqrt{3}$.
- $\sin x = \pm \frac{1}{10}\sqrt{10}$; $\cos x = \mp \frac{3}{10}\sqrt{10}$; $\tan x = -\frac{1}{3}$; $\sec x = \mp \frac{3}{10}\sqrt{10}$;
 $\csc x = \pm \sqrt{10}$.

16. The cosine, the tangent, the cotangent, and the secant are negative when the angle is obtuse.
18. $\sin 90^\circ = 1$, $\cos 90^\circ = 0$, $\cot 90^\circ = 0$, $\sec 90^\circ = \infty$,
 $\csc 90^\circ = 1$;
 $\sin 180^\circ = 0$, $\tan 180^\circ = 0$, $\cot 180^\circ = \infty$, $\sec 180^\circ = -1$,
 $\csc 180^\circ = \infty$;
 $\sin 270^\circ = -1$, $\cos 270^\circ = 0$, $\tan 270^\circ = \infty$, $\sec 270^\circ = \infty$,
 $\csc 270^\circ = -1$;
 $\sin 360^\circ = 0$, $\cos 360^\circ = 1$, $\tan 360^\circ = 0$, $\cot 360^\circ = \infty$,
 $\sec 360^\circ = 1$.
19. $\sin 450^\circ = 1$; $\tan 540^\circ = 0$; $\cos 630^\circ = 0$; $\cot 720^\circ = \infty$;
 $\sin 810^\circ = 1$; $\csc 900^\circ = \infty$.
20. 0. 21. 0. 22. 0. 23. $a^2 - b^2 + 4ab$.

Exercise XIII. Page 51

- | | | |
|---------------------|-------------------------|--------------------------|
| 1. $-\cos 20^\circ$ | 8. $-\sin 24^\circ$ | 15. $-\cos 15^\circ 33'$ |
| 2. $\sin 8^\circ$ | 9. $\cos 1^\circ$ | 16. $\cot 0^\circ 45'$ |
| 3. $-\sin 10^\circ$ | 10. $-\cot 30^\circ$ | 17. $-\cot 40^\circ 43'$ |
| 4. $-\cot 35^\circ$ | 11. $\tan 6^\circ$ | 18. $\csc 29^\circ 45'$ |
| 5. $-\tan 1^\circ$ | 12. $-\csc 26^\circ$ | 19. $\sec 2^\circ 25'$ |
| 6. $-\csc 20^\circ$ | 13. $-\sec 1^\circ$ | |
| 7. $\csc 23^\circ$ | 14. $\sin 16^\circ 11'$ | |
-
- | | |
|---|---|
| 20. $\sin (-75^\circ) = -\cos 15^\circ$;
$\cos (-75^\circ) = \sin 15^\circ$;
$\tan (-75^\circ) = -\cot 15^\circ$;
$\cot (-75^\circ) = -\tan 15^\circ$. | 23. $\sin (-345^\circ) = \sin 15^\circ$;
$\cos (-345^\circ) = \cos 15^\circ$;
$\tan (-345^\circ) = \tan 15^\circ$;
$\cot (-345^\circ) = \cot 15^\circ$. |
| 21. $\sin (-127^\circ) = -\cos 37^\circ$;
$\cos (-127^\circ) = -\sin 37^\circ$;
$\tan (-127^\circ) = \cot 37^\circ$;
$\cot (-127^\circ) = \tan 37^\circ$. | 24. $\sin (-52^\circ 37') = -\cos 37^\circ 23'$;
$\cos (-52^\circ 37') = \sin 37^\circ 23'$;
$\tan (-52^\circ 37') = -\cot 37^\circ 23'$;
$\cot (-52^\circ 37') = -\tan 37^\circ 23'$. |
| 22. $\sin (-200^\circ) = \sin 20^\circ$;
$\cos (-200^\circ) = -\cos 20^\circ$;
$\tan (-200^\circ) = -\tan 20^\circ$;
$\cot (-200^\circ) = -\cot 20^\circ$. | 25. $\sin (-196^\circ 54') = \sin 16^\circ 54'$;
$\cos (-196^\circ 54') = -\cos 16^\circ 54'$;
$\tan (-196^\circ 54') = -\tan 16^\circ 54'$;
$\cot (-196^\circ 54') = -\cot 16^\circ 54'$. |

26. $\sin 120^\circ = +\frac{1}{2}\sqrt{3}$; $\cos 120^\circ = -\frac{1}{2}$; $\tan 120^\circ = -\sqrt{3}$; $\cot 120^\circ = -\frac{1}{\sqrt{3}}$.
27. $\sin 135^\circ = +\frac{1}{2}\sqrt{2}$; $\cos 135^\circ = -\frac{1}{2}\sqrt{2}$; $\tan 135^\circ = -1$; $\cot 135^\circ = -1$.
28. $\sin 150^\circ = \frac{1}{2}$; $\cos 150^\circ = -\frac{1}{2}\sqrt{3}$; $\tan 150^\circ = -\frac{1}{\sqrt{3}}$; $\cot 150^\circ = -\sqrt{3}$.
29. $\sin 210^\circ = -\frac{1}{2}$; $\cos 210^\circ = -\frac{1}{2}\sqrt{3}$; $\tan 210^\circ = +\frac{1}{\sqrt{3}}$; $\cot 210^\circ = +\sqrt{3}$.
30. $\sin 225^\circ = -\frac{1}{2}\sqrt{2}$; $\cos 225^\circ = -\frac{1}{2}\sqrt{2}$; $\tan 225^\circ = 1$; $\cot 225^\circ = 1$.
31. $\sin 240^\circ = -\frac{1}{2}\sqrt{3}$; $\cos 240^\circ = -\frac{1}{2}$; $\tan 240^\circ = +\sqrt{3}$; $\cot 240^\circ = +\frac{1}{\sqrt{3}}$.
32. $\sin 300^\circ = -\frac{1}{2}\sqrt{3}$; $\cos 300^\circ = \frac{1}{2}$; $\tan 300^\circ = -\sqrt{3}$; $\cot 300^\circ = -\frac{1}{\sqrt{3}}$.
33. $\sin(-30^\circ) = -\frac{1}{2}$; $\cos(-30^\circ) = +\frac{1}{2}\sqrt{3}$; $\tan(-30^\circ) = -\frac{1}{\sqrt{3}}$; $\cot(-30^\circ) = -\sqrt{3}$.
34. $\sin(-225^\circ) = +\frac{1}{2}\sqrt{2}$; $\cos(-225^\circ) = -\frac{1}{2}\sqrt{2}$; $\tan(-225^\circ) = -1$; $\cot(-225^\circ) = -1$.
35. $\cos x = -\frac{1}{2}\sqrt{2}$; $\tan x = 1$; $\cot x = 1$; $x = 225^\circ$.
36. $\sin x = \frac{1}{2}$; $\cos x = -\frac{1}{2}\sqrt{3}$; $\tan x = -\frac{1}{\sqrt{3}}$; $x = 150^\circ$.
37. $\sin 3540^\circ = -\frac{1}{2}\sqrt{3}$; $\cos 3540^\circ = \frac{1}{2}$; $\tan 3540^\circ = -\sqrt{3}$; $\cot 3540^\circ = -\frac{1}{\sqrt{3}}$.
38. 210° and 330° ; 120° and 300° .
39. 135° , 225° , and -225° ; 150° and -30° .
40. 30° , 150° , 390° , and 510° .
41. $\sin 168^\circ$; $\cos 334^\circ$; $\tan 225^\circ$; $\cot 252^\circ$;
 $\sin 349^\circ$; $\cos 240^\circ$; $\tan 64^\circ$; $\cot 177^\circ$.
42. 0.8480. 43. -1.9522. 44. $(a-b)\sin x$.
45. $m \sin x \cos x$. 46. 0.
46. $(a-b)\cot x - (a+b)\tan x$. 49. $\cos x \sin y - \sin x \cos y$.
47. $a^2 + b^2 + 2ab \cos x$. 50. $\tan x$.
51. Positive between $x = 0^\circ$ and $x = 135^\circ$, and between $x = 315^\circ$ and $x = 360^\circ$; negative between $x = 135^\circ$ and $x = 315^\circ$.
52. Positive between $x = 45^\circ$ and $x = 225^\circ$; negative between $x = 0^\circ$ and $x = 45^\circ$, and between $x = 225^\circ$ and $x = 360^\circ$.
53. $\sin(x - 90^\circ) = -\cos x$; $\cos(x - 90^\circ) = \sin x$;
 $\tan(x - 90^\circ) = -\cot x$; $\cot(x - 90^\circ) = -\tan x$.
54. $\sin(x - 180^\circ) = -\sin x$; $\cos(x - 180^\circ) = -\cos x$;
 $\tan(x - 180^\circ) = \tan x$; $\cot(x - 180^\circ) = \cot x$.

Exercise XIV. Page 60

1. $\sin(x+y) = \frac{4}{5}$; $\cos(x+y) = \frac{3}{5}$.
2. $\sin(90^\circ - y) = \cos y$; $\cos(90^\circ - y) = \sin y$.
3. $\sin(90^\circ + y) = \cos y$; $\cos(90^\circ + y) = -\sin y$;
 $\tan(90^\circ + y) = -\cot y$; $\cot(90^\circ + y) = -\tan y$.
4. $\sin(180^\circ - y) = \sin y$; $\cos(180^\circ - y) = -\cos y$;
 $\tan(180^\circ - y) = -\tan y$; $\cot(180^\circ - y) = -\cot y$.
5. $\sin(180^\circ + y) = -\sin y$; $\cos(180^\circ + y) = -\cos y$;
 $\tan(180^\circ + y) = \tan y$; $\cot(180^\circ + y) = \cot y$.
6. $\sin(270^\circ - y) = -\cos y$; $\cos(270^\circ - y) = -\sin y$;
 $\tan(270^\circ - y) = \cot y$; $\cot(270^\circ - y) = \tan y$.
7. $\sin(270^\circ + y) = -\cos y$; $\cos(270^\circ + y) = \sin y$;
 $\tan(270^\circ + y) = -\cot y$; $\cot(270^\circ + y) = -\tan y$.
8. $\sin(360^\circ - y) = -\sin y$; $\cos(360^\circ - y) = \cos y$;
 $\tan(360^\circ - y) = -\tan y$; $\cot(360^\circ - y) = -\cot y$.
9. $\sin(360^\circ + y) = \sin y$; $\cos(360^\circ + y) = \cos y$;
 $\tan(360^\circ + y) = \tan y$; $\cot(360^\circ + y) = \cot y$.
10. $\sin(x - 90^\circ) = -\cos x$; $\cos(x - 90^\circ) = \sin x$;
 $\tan(x - 90^\circ) = -\cot x$; $\cot(x - 90^\circ) = -\tan x$.
11. $\sin(x - 180^\circ) = -\sin x$; $\cos(x - 180^\circ) = -\cos x$;
 $\tan(x - 180^\circ) = \tan x$; $\cot(x - 180^\circ) = \cot x$.
12. $\sin(x - 270^\circ) = \cos x$; $\cos(x - 270^\circ) = -\sin x$;
 $\tan(x - 270^\circ) = -\cot x$; $\cot(x - 270^\circ) = -\tan x$.
13. $\sin(-y) = -\sin y$; $\cos(-y) = \cos y$;
 $\tan(-y) = -\tan y$; $\cot(-y) = -\cot y$.
14. $\sin(45^\circ - y) = \frac{1}{2}\sqrt{2}(\cos y - \sin y)$; $\cos(45^\circ - y) = \frac{1}{2}\sqrt{2}(\cos y + \sin y)$;
 $\tan(45^\circ - y) = \frac{1 - \tan y}{1 + \tan y}$; $\cot(45^\circ - y) = \frac{\cot y + 1}{\cot y - 1}$.
15. $\sin(45^\circ + y) = \frac{1}{2}\sqrt{2}(\cos y + \sin y)$; $\cos(45^\circ + y) = \frac{1}{2}\sqrt{2}(\cos y - \sin y)$;
 $\tan(45^\circ + y) = \frac{1 + \tan y}{1 - \tan y}$; $\cot(45^\circ + y) = \frac{\cot y - 1}{\cot y + 1}$.
16. $\sin(30^\circ + y) = \frac{1}{2}(\cos y + \sqrt{3}\sin y)$; $\cos(30^\circ + y) = \frac{1}{2}(\sqrt{3}\cos y - \sin y)$;
 $\tan(30^\circ + y) = \frac{\frac{1}{2}\sqrt{3} + \tan y}{1 - \frac{1}{2}\sqrt{3}\tan y}$; $\cot(30^\circ + y) = \frac{\sqrt{3}\cot y - 1}{\cot y + \sqrt{3}}$.

$$17. \sin(60^\circ - y) = \frac{1}{2}(\sqrt{3} \cos y - \sin y); \cos(60^\circ - y) = \frac{1}{2}(\cos y + \sqrt{3} \sin y);$$

$$\tan(60^\circ - y) = \frac{\sqrt{3} - \tan y}{1 + \sqrt{3} \tan y}; \cot(60^\circ - y) = \frac{\frac{1}{3}\sqrt{3} \cot y + 1}{\cot y - \frac{1}{3}\sqrt{3}}.$$

$$18. 3 \sin x - 4 \sin^3 x. \quad 19. 4 \cos^2 x - 3 \cos x. \quad 20. 0. \quad 21. \frac{1}{2} \sqrt{3}.$$

$$22. \sin \frac{1}{2} x = \sqrt{\frac{1 - 0.4 \sqrt{6}}{2}} = 0.10051; \cos \frac{1}{2} x = \sqrt{\frac{1 + 0.4 \sqrt{6}}{2}} = 0.99493.$$

$$23. \cos 2x = -\frac{1}{2}, \tan 2x = -\sqrt{3}.$$

$$24. \sin 22\frac{1}{2}^\circ = \frac{1}{2} \sqrt{2 - \sqrt{2}} = 0.3827; \cos 22\frac{1}{2}^\circ = \frac{1}{2} \sqrt{2 + \sqrt{2}} = 0.9239;$$

$$\tan 22\frac{1}{2}^\circ = \sqrt{2} - 1 = 0.4142; \cot 22\frac{1}{2}^\circ = \sqrt{2} + 1 = 2.4142.$$

$$25. \sin 15^\circ = \frac{1}{4} \sqrt{2 - \sqrt{3}} = 0.2588; \cos 15^\circ = \frac{1}{4} \sqrt{2 + \sqrt{3}} = 0.9659;$$

$$\tan 15^\circ = 2 - \sqrt{3} = 0.2679; \cot 15^\circ = 2 + \sqrt{3} = 3.7321.$$

$$34. \sin A + \sin B + \sin C = \sin A + \sin B + \sin [180^\circ - (A + B)]$$

$$= \sin A + \sin B + \sin (A + B)$$

By [20] and [12],

$$= 2 \sin \frac{1}{2} (A + B) \cos \frac{1}{2} (A - B) + 2 \sin \frac{1}{2} (A + B) \cos \frac{1}{2} (A + B)$$

$$= 2 \sin \frac{1}{2} (A + B) [\cos \frac{1}{2} (A - B) + \cos \frac{1}{2} (A + B)]$$

By [22],

$$= 2 \sin \frac{1}{2} (A + B) (2 \cos \frac{1}{2} A \cos \frac{1}{2} B)$$

$$= 4 \sin \frac{1}{2} (A + B) \cos \frac{1}{2} A \cos \frac{1}{2} B.$$

$$\text{But } \cos \frac{1}{2} C = \cos [90^\circ - \frac{1}{2} (A + B)] = \sin \frac{1}{2} (A + B).$$

$$\therefore \sin A + \sin B + \sin C = 4 \cos \frac{1}{2} A \cos \frac{1}{2} B \cos \frac{1}{2} C.$$

35. Proof similar to that for 34.

$$38. \frac{2}{\sin 2x}.$$

$$42. \tan^2 x.$$

$$46. \frac{\cos(x+y)}{\sin x \sin y}.$$

$$39. 2 \cot 2x.$$

$$43. \frac{\cos(x-y)}{\cos x \cos y}.$$

$$47. \tan x \tan y.$$

$$40. \frac{\cos(x-y)}{\sin x \cos y}.$$

$$44. \frac{\cos(x+y)}{\cos x \cos y}.$$

$$41. \frac{\cos(x+y)}{\sin x \cos y}.$$

$$45. \frac{\cos(x-y)}{\sin x \sin y}.$$

Exercise XV. Page 63

$$1. \sin^{-1} \frac{1}{2} \sqrt{3} = 60^\circ + 2n\pi \text{ or } 120^\circ + 2n\pi;$$

$$\tan^{-1} \frac{1}{2} \sqrt{3} = 30^\circ + 2n\pi \text{ or } 210^\circ + 2n\pi;$$

$$\text{vers}^{-1} \frac{1}{2} = 60^\circ + 2n\pi \text{ or } 300^\circ + 2n\pi;$$

$$\cos^{-1}(-\frac{1}{2}\sqrt{2}) = 135^\circ + 2n\pi \text{ or } 225^\circ + 2n\pi;$$

$$\csc^{-1}\sqrt{2} = 45^\circ + 2n\pi \text{ or } 135^\circ + 2n\pi;$$

$$\tan^{-1}\infty = 90^\circ + 2n\pi \text{ or } 270^\circ + 2n\pi;$$

$$\sec^{-1}2 = 60^\circ + 2n\pi \text{ or } 300^\circ + 2n\pi;$$

$$\cos^{-1}(-\frac{1}{2}\sqrt{3}) = 150^\circ + 2n\pi \text{ or } 210^\circ + 2n\pi.$$

4. $\frac{1}{2}\sqrt{2}$. 10. $\pm \frac{1}{13}$. 12. $\pm \frac{1}{2}\sqrt{2}$.
 8. $0^\circ, 90^\circ, 180^\circ$. 11. $\pm \frac{7}{14}$. 13. $x = 0$ or $\pm \frac{1}{2}\sqrt{3}$.

Exercise XVI. Page 67

1. If, for instance, $C = 90^\circ$, [25] becomes $\frac{a}{c} = \sin A$.
 3. $a^2 = b^2 + c^2$; $a^2 = b^2 + c^2 - 2bc$; $a^2 = b^2 + c^2 + 2bc$; a right triangle;
 a straight line; a straight line.
 4. $b = a \cos C + c \cos A$; $a = b \cos C + c \cos B$; $c = b \cos A$.
 6. 90° .
 7. (i) $\frac{a-b}{a+b} = \tan(A - 45^\circ)$; a right triangle.
 (ii) $a+b = (a-b)(2 + \sqrt{3})$; an isosceles triangle with the angles 30° ,
 30° , 120° .

Exercise XVII. Page 69

9. 300 yards. 15. $a = 5$; $c = 9.6593$.
 10. $AB = 59.564$ miles; 16. $a = 7$; $b = 8.573$.
 $AC = 54.285$ miles. 17. Sides, 600 feet and 1039.2 feet;
 11. 4.6064 miles; 4.4494 miles; altitude, 519.6 feet.
 3.7733 miles. 18. 855 : 1607.
 12. 4.1501 and 8.67. 19. 5.438 and 6.857.
 13. 6.1433 miles and 8.7918 miles. 20. 15.588.
 14. 8 and 5.4723.

Exercise XVIII. Page 74

1. Two; one; no solution; one; two; no solution; one.
 11. 420. 12. 124.617.

Exercise XIX. Page 78

- | | | |
|-------------|-------------------|---|
| 11. 6. | 15. 25. | 18. 10.266 miles. |
| 12. 10.392. | 16. 3800 yards. | 19. 5.0032 and 2.3385. |
| 14. 8.9212. | 17. 729.67 yards. | 20. $26^{\circ} 0' 10''$ and $14^{\circ} 5' 50''$. |
| | 21. 430.85 yards. | |

Exercise XX. Page 83

- | | |
|---|---|
| 11. $A = 36^{\circ} 52' 12''$; $B = 53^{\circ} 7' 48''$; $C = 90^{\circ}$. | |
| 12. $A = B = 33^{\circ} 33' 27''$; $C = 112^{\circ} 53' 6''$. | |
| 13. $A = B = C = 60^{\circ}$. | |
| 14. $A = 28^{\circ} 57' 18''$; $B = 46^{\circ} 34' 6''$; $C = 104^{\circ} 28' 38''$. | |
| 15. $A = 45^{\circ}$; $B = 120^{\circ}$; $C = 15^{\circ}$. | 21. 54.516 miles. |
| 16. $A = 45^{\circ}$; $B = 60^{\circ}$; $C = 75^{\circ}$. | 22. $84^{\circ} 14' 34''$. |
| 17. $4^{\circ} 23' 2''$ W. of N., or W. of S. | 23. $54^{\circ} 48' 54''$. |
| 18. 60° . | 24. 105° ; 15° ; 60° . |
| 20. 0.88877. | 25. 12.434 inches. |

Exercise XXI. Page 87

- | | | |
|---------------|----------------------|---|
| 1. 4,338,600. | 6. 26,208. | 11. 0.19975. |
| 2. 365.68. | 7. 15,540. | 12. $F = ab \sin A$. |
| 3. 13,260. | 8. 29,450 or 6982.8. | 13. $F = \frac{1}{4}(a^2 - b^2) \tan A$. |
| 4. 8160. | 9. 17.3206. | 14. 2,421,000. |
| 5. 240. | 10. 10.392. | 15. 30° ; 30° ; 120° . |

Exercise XXII. Page 88

- | | |
|--------------------------------|-----------------------|
| 1. 21.166 miles; 24.966 miles. | 4. 30° . |
| 2. 6.3399 miles. | 5. 20 feet. |
| 3. 119.29 feet. | 6. 2.6247 or 21.4587. |

Exercise XXIII. Page 90

- | | | |
|---------------------------------|---------------------------|-------------------|
| 1. 106.70 feet;
142.86 feet. | 3. $37^{\circ} 34' 5''$. | 6. 2922.4 miles. |
| 2. 1023.9 feet. | 4. 238,410 miles. | 7. 60° . |
| | 5. 861,860 miles. | 8. 3.2068. |

9. 6.6031.
 10. 199.56 feet.
 11. 43.107 feet.
 12. 45 feet.
 13. $26^{\circ} 34'$.
 14. 78.367 feet.
 15. 75 feet.
 16. 1.4446 miles.
 17. 7912.8 miles.
 18. 56.649 feet.
 19. 69.282 feet.
 20. 260.21 feet;
 8690.3 feet.
 21. 1.3438 miles.
 22. 235.81 yards.
 26. 8.0076 inches.
 29. 460.46 feet.
 30. 88.936 feet.
 31. 13.657 miles per
 hour.
 33. 56.564 feet.
 34. 51.595 feet.
 35. 101.892 feet.
 37. N. $76^{\circ} 56' E.$;
 13.938 miles per
 hour.
 38. 442.11 yards.
 39. 255.78 feet.
 40. 3121.1 feet;
 3633.5 feet.
 41. 529.49 feet.
 42. 41.411 feet.
 43. 234.51 feet.
 44. 25.433 miles.
 45. 294.69 feet.
 46. 12,492.6 feet.
 47. 6.3397 miles.
 48. 210.44 feet.
 50. 757.50 feet.
 51. 520.01 yards.
 52. 1366.4 feet.
 53. 658.36 pounds;
 $22^{\circ} 23' 47''$ with
 first force.
 54. 88.326 pounds;
 $45^{\circ} 37' 16''$ with
 known force.
 57. 536.28; 500.16.
 58. 345.48 feet.
 59. 345.46 yards.
 60. 61.23 feet.
 62. 307.77 yards.
 63. 19.8; 35.7; 44.5.
 64. 45° , 135° , 225° ,
 or 315° .
 65. $\cos A = \frac{-m \pm \sqrt{m^2 + 4(n+1)}}{2}$.
 66. $\sin A = \sqrt{\frac{m^2 - n^2}{1 - n^2}}$; $\cos B = \frac{n}{m} \sqrt{\frac{1 - m^2}{1 - n^2}}$.
 67. 60° , 120° , 240° , or 300° .
 68. 0° , 60° , 180° , or 300° .
 69. 0° , 30° , 150° , 180° , 210° , 330° .
 71. $r = \frac{a}{2} \cot \frac{180^{\circ}}{n}$; $R = \frac{a}{2} \csc \frac{180^{\circ}}{n}$.
 72. $F = \frac{1}{2} bc \sin A$.
 73. $F = \frac{1}{2} c^2 \sin A \sin B \csc (A+B)$.
 74. $F = \sqrt{s(s-a)(s-b)(s-c)}$.
 76. 199 acres 8 square chains.
 77. 210 acres 9.1 square chains.
 78. 12 acres 9.78 square chains.
 79. 3 acres 0.392 square chains.
 80. 12 acres 3.45 square chains.
 81. 4 acres 6.634 square chains.
 82. 14 acres 5.54 square chains.
 83. 61 acres 4.97 square chains.
 84. 4 acres 6.633 square chains.
 85. 13.93 chains; 23.21 chains;
 32.50 chains.
 86. 9 acres 0.055 square chains.
 88. 876.34.
 89. 1229.5.
 91. 1075.3.
 92. 2660.4.
 93. 16,281.
 94. 435.76 square feet.
 95. 49,088 square feet.
 96. 749.95 square feet.
 97. 422.38 square feet.

98. 1834.95 square feet. 108. 6086.4 feet. 111. 228.98 miles;
 feet. 109. $5^{\circ} 25' 6''$ S.; $11^{\circ} 39' 6''$ S.
 99. 26.88. 457.49 miles. 112. S. $56^{\circ} 7' 32''$ E.;
 102. 6. 110. 460.79 miles; 202.58 miles.
 107. 6. 383.13 miles.
 113. N. $17^{\circ} 25' 22''$ W.; 119. $33^{\circ} 18' 22''$ N.; $36^{\circ} 23' 53''$ W.
 $37^{\circ} 46' 13''$ N. 120. N. $28^{\circ} 47' 26''$ E.; 1292.8 miles.
 114. 244.35 miles; S. $56^{\circ} 10' 49''$ E. 121. S. $50^{\circ} 39' 44''$ W.;
 250.84 miles; $20^{\circ} 9' 30''$ W.
 115. 359.87 miles. 122. $38^{\circ} 20' 34''$ N.; $55^{\circ} 12' 4''$ W.
 117. Long. $68^{\circ} 54' 39''$ W. 123. 171.14 miles; $32^{\circ} 43' 38''$ W.
 118. 103.57 miles.
 124. N. $36^{\circ} 52' 12''$ W.; $36^{\circ} 7' 37''$ W.
 125. 173.18 miles; $51^{\circ} 16' 16''$ S.; $34^{\circ} 12' 43''$ E.
 126. S. $50^{\circ} 57' 48''$ E.; $47^{\circ} 14' 35''$ N.; $20^{\circ} 48' 37''$ W.
 127. N. $53^{\circ} 20' 21''$ E., $16^{\circ} 6' 57''$ W.; or N. $53^{\circ} 20' 21''$ W., $25^{\circ} 53' 3''$ W.
 128. N. $47^{\circ} 42' 33''$ E., $19^{\circ} 27' 22''$ N., $121^{\circ} 50' 34''$ E.; or N. $47^{\circ} 42' 33''$ W.,
 $19^{\circ} 27' 22''$ N., $116^{\circ} 9' 26''$ E.; or S. $47^{\circ} 42' 33''$ E., $14^{\circ} 32' 38''$ N.,
 $121^{\circ} 48' 20''$ E.; or S. $47^{\circ} 42' 33''$ W., $14^{\circ} 32' 38''$ N., $116^{\circ} 11' 40''$ E.
 129. 359.82 miles; 359.73 miles; 359.50 miles.
 130. $35^{\circ} 49' 10''$ S., $22^{\circ} 2' 44''$ W.; N. $61^{\circ} 42'$ W.; 183.16 miles.
 131. $42^{\circ} 15' 29''$ N., $69^{\circ} 5' 11''$ W.; N. $72^{\circ} 32' 40''$ E.; 44.939 miles.
 132. $32^{\circ} 53' 34''$ S., $13^{\circ} 1' 53''$ E.; N. $72^{\circ} 3' 43''$ W.; 287.16 miles.

Exercise XXIV. Page 107

(The solutions here given are for angles less than 360° .)

79. $\sin \frac{1}{2}x = \pm \frac{1}{2}\sqrt{5}$; $\cos \frac{1}{2}x = \pm \frac{2}{3}\sqrt{5}$. 81. $\pm \frac{1}{2}\sqrt{3}$.
 80. $\pm \sqrt{5} - 2$. 82. $\pm \frac{2}{3}$ or $\pm \frac{4}{3}$.
 83. $\pm 2\sqrt{2}$, $\pm \frac{1}{2}(9\sqrt{3} + 8\sqrt{2})$, or $\pm \frac{1}{2}(9\sqrt{3} - 8\sqrt{2})$.
 84. $\pm \frac{1}{2}$. 85. $\frac{1}{2}(\sqrt{5} - 1)$; $\frac{1}{2}(\sqrt{5} + 1)$.
 86. $(a^3 + b^3)^{\frac{1}{3}}$. 91. $\pm \frac{a+1}{\sqrt{2a+1}}$. 96. $\tan^{-1} \frac{2x}{1-2x^2}$.
 87. $\left(\frac{1+m}{2}\right)^{\frac{1}{2}}(1 \mp 2m)$. 92. 4. 97. 2.
 88. $\pm \frac{1}{2}\sqrt{2}$ or $\pm \frac{1}{2}\sqrt{3}$. 93. $\tan(x+y)$. 98. $-\tan^2 x + \cot^2 x$.
 89. $\frac{1}{2}$. 94. $\frac{\sin x}{\sin y}$. 99. $x = \frac{1}{2}\pi$ or $\frac{3}{2}\pi$.
 90. $\frac{1}{2}$ or $-\frac{3}{2}$. 95. $-\tan x$. 100. $x = 90^{\circ}$ or 270° .

101. $x = 21^\circ 28'$ or $158^\circ 32'$.
 102. $x = 0^\circ$ or 90° .
 103. $x = 30^\circ, 150^\circ, 199^\circ 28',$ or $340^\circ 32'$.
 104. $x = 51^\circ 19', 180^\circ,$ or $308^\circ 41'$.
 105. $x = 0^\circ, 120^\circ, 180^\circ,$ or 240° .
 106. $x = 45^\circ, 161^\circ 34', 225^\circ,$ or $341^\circ 34'$.
 107. $\theta = 60^\circ, 120^\circ, 240^\circ,$ or 300° .
 108. $\theta = 26^\circ 34'$ or $206^\circ 34'$. 110. $x = 45^\circ$ or 135° .
 109. $x = 30^\circ$ or 150° . 111. $x = 30^\circ, 150^\circ,$ or 270° .
 112. $x = 35^\circ 16'', 144^\circ 44', 215^\circ 16',$ or $324^\circ 44'$.
 113. $x = 75^\circ 58'$ or $255^\circ 58'$.
 114. $\theta = 60^\circ, 180^\circ,$ or 300° . 116. $x = 30^\circ, 150^\circ, 210^\circ,$ or 330° .
 115. $\theta = 90^\circ$ or $143^\circ 8'$. 117. $x = 30^\circ, 150^\circ,$ or 270° .
 118. $x = 26^\circ 34', 90^\circ, 206^\circ 34',$ or 270° .
 119. $x = 45^\circ, 135^\circ, 225^\circ,$ or 315° .
 120. $x = 45^\circ, 135^\circ, 225^\circ,$ or 315° .
 121. $x = 15^\circ, 75^\circ, 135^\circ, 195^\circ, 255^\circ,$ or 315° .
 122. $z = 45^\circ, 135^\circ, 225^\circ,$ or 315° .
 123. $x = 0^\circ, 60^\circ, 120^\circ, 180^\circ, 240^\circ,$ or 300° .
 124. $x = 27^\circ 58', 135^\circ, 242^\circ 2',$ or 315° .
 125. $x = 0^\circ, 45^\circ, 180^\circ,$ or 225° .
 126. $x = 32^\circ 46', 147^\circ 14', 212^\circ 46',$ or $327^\circ 14'$.
 127. $x = 0^\circ, 45^\circ, 90^\circ, 180^\circ, 225^\circ,$ or 270° .
 128. $x = 0^\circ, 65^\circ 42', 180^\circ,$ or $204^\circ 18'$.
 129. $x = 0^\circ, 90^\circ, 120^\circ, 240^\circ,$ or 270° .
 130. $x = 0^\circ, 36^\circ, 72^\circ, 108^\circ, 144^\circ, 180^\circ, 216^\circ, 252^\circ, 288^\circ,$ or 324° .
 131. $x = 30^\circ, 150^\circ, 210^\circ,$ or 330° .
 132. $x = 60^\circ$ or 240° .
 133. $x = 54^\circ 44', 125^\circ 16', 234^\circ 44',$ or $305^\circ 16'$.
 134. $x = 105^\circ$ or 345° .
 135. $x = \tan^{-1} \frac{a^2 - 1}{2a}$.
 136. $x = \cos^{-1} \frac{-a \pm \sqrt{a^2 + 8a + 8}}{4}$.

137. $x = 135^\circ, 315^\circ$, or $\frac{1}{2} \sin^{-1}(1 - a)$.
 138. $x = 30^\circ, 60^\circ, 120^\circ, 150^\circ, 210^\circ, 240^\circ, 300^\circ$, or 330° .
 139. $x = 60^\circ, 90^\circ, 120^\circ, 240^\circ, 270^\circ$, or 300° .
 140. $x = 60^\circ, 90^\circ, 120^\circ, 240^\circ, 270^\circ$, or 300° .
 141. $x = 120^\circ$. 142. $x = 14^\circ 29', 30^\circ, 150^\circ$, or $165^\circ 31'$.
 143. $x = 60^\circ, 90^\circ, 270^\circ$, or 300° .
 144. $x = 0^\circ, 20^\circ, 100^\circ, 140^\circ, 180^\circ, 220^\circ, 260^\circ$, or 340° .
 145. $x = 45^\circ, 90^\circ, 135^\circ, 225^\circ, 270^\circ$, or 315° .
 146. $x = 30^\circ, 60^\circ, 90^\circ, 120^\circ, 150^\circ, 210^\circ, 240^\circ, 270^\circ, 300^\circ$, or 330° .
 147. $x = 0^\circ, 45^\circ, 90^\circ, 180^\circ, 225^\circ$, or 270° .
 148. $x = 30^\circ, 60^\circ, 120^\circ, 150^\circ, 210^\circ, 240^\circ, 300^\circ$, or 330° .
 149. $x = 30^\circ, 90^\circ, 150^\circ, 210^\circ, 270^\circ$, or 330° .
 150. $x = 0^\circ, 45^\circ, 180^\circ$, or 225° .
 151. $x = 45^\circ, 60^\circ, 120^\circ, 135^\circ, 225^\circ, 240^\circ, 300^\circ$, or 315° .
 152. $x = 0^\circ, 45^\circ, 135^\circ, 225^\circ$, or 315° .
 153. $x = 30^\circ, 90^\circ, 150^\circ, 210^\circ, 270^\circ$, or 330° .
 154. $x = 8^\circ$ or 168° .
 155. $x = 40^\circ 12', 139^\circ 48', 220^\circ 12'$, or $319^\circ 48'$.
 156. $x = 30^\circ$ or 330° .
 157. $x = 60^\circ, 120^\circ, 240^\circ$, or 300° .
 158. $x = 30^\circ, 60^\circ, 120^\circ, 150^\circ, 210^\circ, 240^\circ, 300^\circ$, or 330° .
 159. $x = 53^\circ 8', 126^\circ 52', 233^\circ 8'$, or $306^\circ 52'$.
 160. $x = 30^\circ$. 161. $x = 22^\circ 37'$ or $143^\circ 8'$.
 162. $x = 0^\circ, 20^\circ, 40^\circ, 60^\circ, 80^\circ, 100^\circ, 120^\circ, 140^\circ, 160^\circ, 180^\circ, 200^\circ, 220^\circ, 240^\circ, 260^\circ, 280^\circ, 300^\circ, 320^\circ$, or 340° .
 163. $x = 22^\circ 30', 45^\circ, 67^\circ 30', 90^\circ, 112^\circ 30', 135^\circ, 157^\circ 30', 202^\circ 30', 225^\circ, 247^\circ 30', 270^\circ, 292^\circ 30', 315^\circ$, or $337^\circ 30'$.
 164. $x = 45^\circ$ or 225° . 169. $x = 1$.
 165. $x = \pm 1$ or $\pm \frac{1}{2} \sqrt{21}$. 170. $x = 0$ or ± 1 .
 166. $x = \frac{1}{2} \sqrt{3}$ or $-\frac{1}{2} \sqrt{3}$. 171. $x = \pm \frac{1}{2} \sqrt{2}$.
 167. $x = \frac{1}{2} \sqrt{3}$. 172. $x = \frac{1}{2} \sqrt{3}$.
 168. $x = \frac{1}{2}$. 173. $\theta = 120^\circ$ or 240° .
 174. $x = 60^\circ, 120^\circ, 240^\circ$, or 300° .
 175. $x = 0^\circ, 45^\circ, 135^\circ, 225^\circ$, or 315° .

176. $x = 0^\circ, 180^\circ, 220^\circ 39', \text{ or } 319^\circ 21'.$
 177. $x = 0^\circ, 60^\circ, 120^\circ, 180^\circ, 240^\circ, \text{ or } 300^\circ.$
 178. $\theta = 18^\circ, 90^\circ, 162^\circ, 234^\circ, 270^\circ, \text{ or } 306^\circ.$
 179. $x = 0^\circ, 30^\circ, 90^\circ, 150^\circ, 180^\circ, 210^\circ, 270^\circ, \text{ or } 330^\circ.$
 180. $x = 0^\circ, 90^\circ, 120^\circ, 180^\circ, 240^\circ, \text{ or } 270^\circ.$
 181. $\theta = 0^\circ, 74^\circ 5', 127^\circ 25', 180^\circ, 232^\circ 35', \text{ or } 285^\circ 55'.$
 182. $x = 0^\circ, 90^\circ, 180^\circ, \text{ or } 270^\circ.$
 183. $x = 0^\circ, 45^\circ, 90^\circ, 180^\circ, 225^\circ, \text{ or } 270^\circ.$
 184. $x = 0^\circ, 45^\circ, 120^\circ, 135^\circ, 180^\circ, 225^\circ, 240^\circ, \text{ or } 315^\circ.$
 185. $\theta = 10^\circ 12', 34^\circ 48', 190^\circ 12', \text{ or } 214^\circ 48'.$ 187. $\theta = 29^\circ 19', 105^\circ 41', 209^\circ 19', \text{ or } 285^\circ 41'.$
 186. $x = 90^\circ \text{ or } 270^\circ.$ 188. $x = 1.$
 189. $x = \frac{b \sin \beta - a \cos \beta}{\sin(\beta - \alpha)}; y = \frac{a \cos \alpha - b \sin \alpha}{\sin(\beta - \alpha)}.$
 190. $x = \tan^{-1} \frac{a}{b} + \cos^{-1} \frac{1}{2} \sqrt{a^2 + b^2}; y = \tan^{-1} \frac{a}{b} - \cos^{-1} \frac{1}{2} \sqrt{a^2 + b^2}.$
 191. $\theta = \tan^{-1} \frac{a}{b}; r = \sqrt{a^2 + b^2}.$
 192. $r = \frac{a}{\sin\left(\tan^{-1} \frac{a \cos \beta - b \sin \alpha}{a \sin \beta + b \cos \alpha} + \alpha\right)} = \frac{b}{\cos\left(\tan^{-1} \frac{a \cos \beta - b \sin \alpha}{a \sin \beta + b \cos \alpha} + \beta\right)};$
 $\theta = \tan^{-1} \frac{a \cos \beta - b \sin \alpha}{a \sin \beta + b \cos \alpha}.$
 193. $r = \sqrt{a^2 + b^2 + c^2}; \theta = \tan^{-1} \frac{a}{b}; \phi = \tan^{-1} \frac{c}{\sqrt{a^2 + b^2}}.$
 194. $x = 100; y = 200.$ 195. $x = 76^\circ 10'; y = 15^\circ 30'.$
 196. $r = 225.12, \theta = 24^\circ 13'; \text{ or } r = -225.12, \theta = 204^\circ 13'.$
 197. $r = 151, \theta = 42^\circ 28'; \text{ or } r = -151, \theta = 222^\circ 28'.$
 198. $r = 108, \phi = 120^\circ, \theta = 330^\circ; r = 108, \phi = 300^\circ, \theta = 210^\circ;$
 $r = -108, \phi = 120^\circ, \theta = 150^\circ; \text{ or } r = -108, \phi = 300^\circ, \theta = 30^\circ.$
 199. $x = r \operatorname{vers}^{-1}\left(\frac{y}{r}\right) \mp \sqrt{2ry - y^2}.$

Exercise XXV. Page 121

1. $\log_{10} 6 = 0.77815; \log_{10} 14 = 1.14613; \log_{10} 21 = 1.32222;$
 $\log_{10} 4 = 0.60206; \log_{10} 12 = 1.07918; \log_{10} 5 = 0.69897;$
 $\log_{10} \frac{1}{2} = \bar{1}.69897; \log_{10} \frac{1}{4} = \bar{1}.39794; \log_{10} \frac{1}{5} = \bar{1}.89086;$
 $\log_{10} \frac{1}{10} = 0.02119.$

2. $\log_3 10 = 3.3219$; $\log_2 5 = 2.3219$; $\log_3 5 = 1.4650$;
 $\log_7 \frac{1}{2} = -0.3562$; $\log_5 \frac{2}{3} = -2.2620$.
3. $\log_e 2 = 0.69315$; $\log_e 3 = 1.09862$; $\log_e 5 = 1.60945$;
 $\log_e 7 = 1.94593$; $\log_e 8 = 2.07943$; $\log_e 9 = 2.19724$;
 $\log_e \frac{2}{3} = -0.40547$; $\log_e \frac{4}{3} = -0.22315$; $\log_e \frac{5}{3} = 0.25952$;
 $\log_e \frac{7}{5} = -2.14845$.
4. $x = 1.54396$; $x = 0.83048$; $x = 0.42062$.

Exercise XXVI. Page 126

1. $\log_e 3 = 1.09861$. 2. $\log_e 5 = 1.60944$. 3. $\log_e 7 = 1.94591$.
 4. $\log_e 10 = 2.3025850930$.
 5. $\log_{10} 2 = 0.30103$; $\log_{10} e = 0.43429$; $\log_{10} 11 = 1.04139$.

Exercise XXVII. Page 128

1. $\sin 1' = 0.00029088820$; $\cos 1' = 0.99999995769$;
 $\tan 1' = 0.000290888212$.
 2. $\sin 2' = 0.000581776$. 3. $\sin 1^\circ = 0.0175$. 6. $0^\circ 40' 9''$.

Exercise XXVIII. Page 130

1. $\sin 6' = 0.0017453$; $\cos 6' = 0.9999985$.
 2. $\sin 2^\circ = 0.034902$; $\cos 2^\circ = 0.999392$.
 3. $\sin 3^\circ = 0.052340$; $\cos 3^\circ = 0.998632$.
 4. $\sin 4^\circ = 0.069762$; $\cos 4^\circ = 0.997568$.
 5. $\sin 5^\circ = 0.087163$; $\cos 5^\circ = 0.996201$.

Exercise XXIX. Page 135

1. The 6 sixth roots of -1 are:

$$\frac{\sqrt{3} + i}{2}, i, \frac{-\sqrt{3} + i}{2}, \frac{-\sqrt{3} - i}{2}, -i, \frac{\sqrt{3} - i}{2}.$$

The 6 sixth roots of $+1$ are:

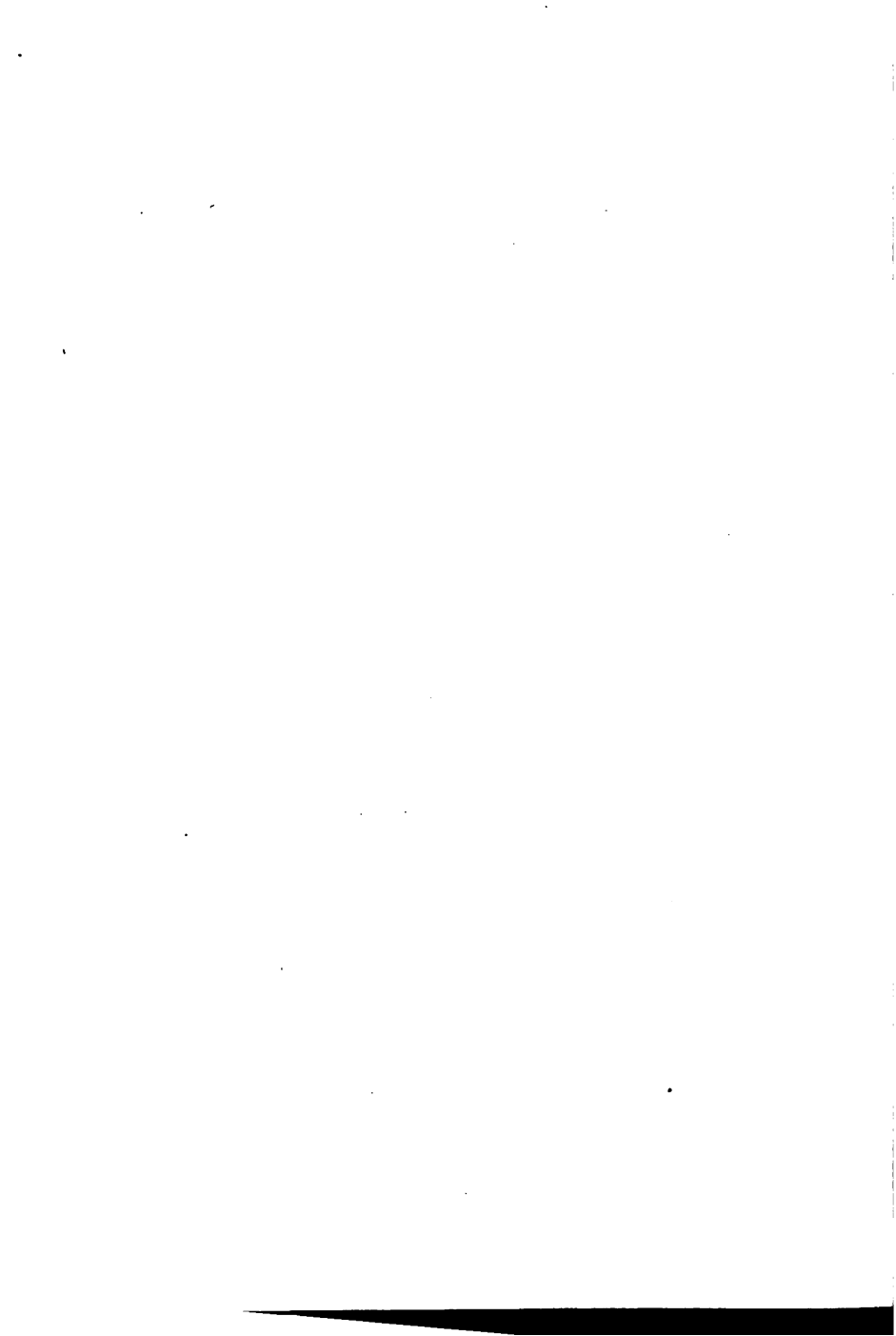
$$1, \frac{1 + \sqrt{-3}}{2}, \frac{-1 + \sqrt{-3}}{2}, -1, \frac{-1 - \sqrt{-3}}{2}, \frac{1 - \sqrt{-3}}{2}.$$

2. $\frac{\sqrt{3} + i}{2}, \frac{-\sqrt{3} + i}{2}, -i.$

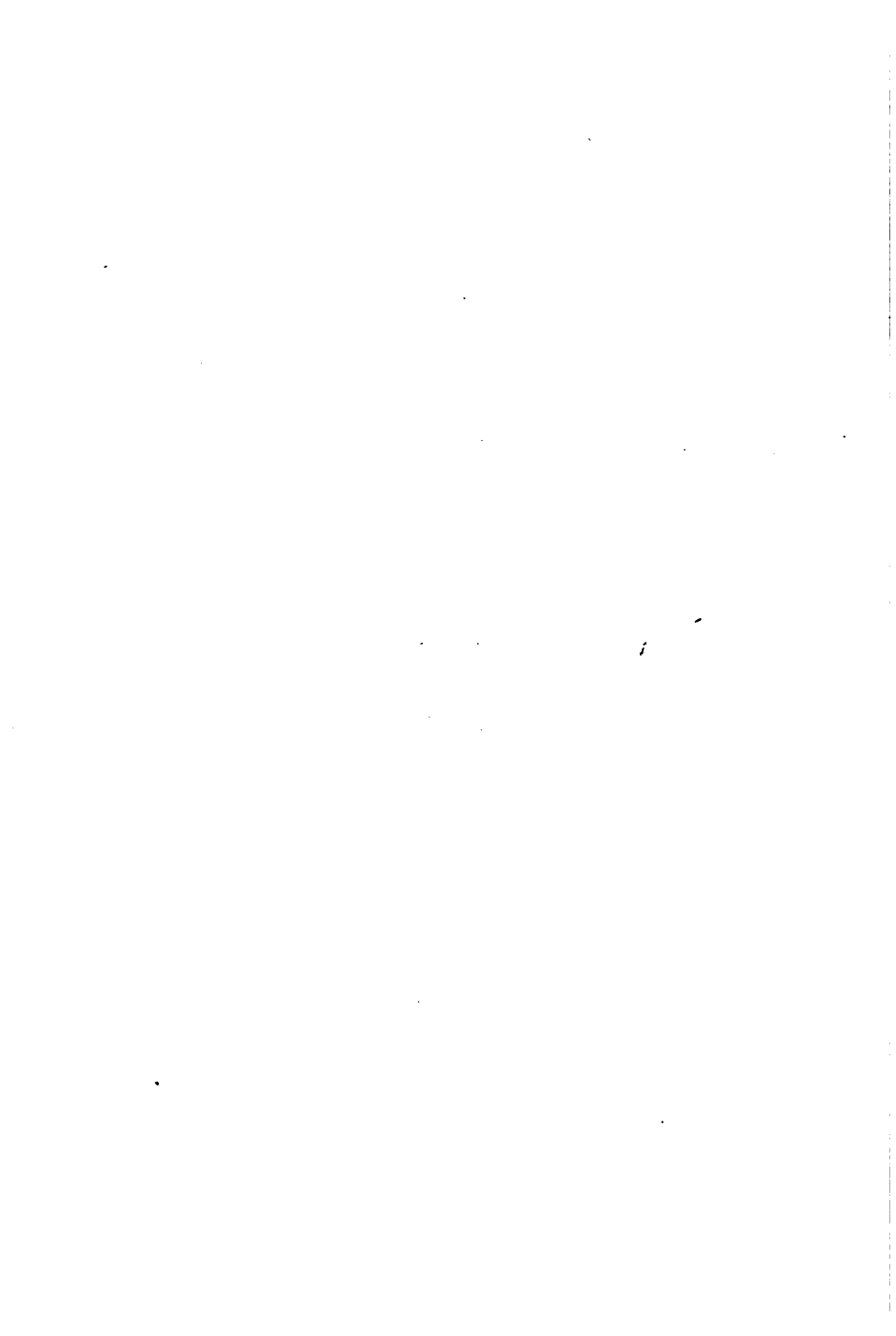
3. $\cos 67\frac{1}{2}^\circ + i \sin 67\frac{1}{2}^\circ$, $\cos 157\frac{1}{2}^\circ + i \sin 157\frac{1}{2}^\circ$, $\cos 247\frac{1}{2}^\circ + i \sin 247\frac{1}{2}^\circ$,
 $\cos 337\frac{1}{2}^\circ + i \sin 337\frac{1}{2}^\circ$.
4. $\sin 4\theta = 4 \cos^3 \theta \sin \theta - 4 \cos \theta \sin^3 \theta$;
 $\cos 4\theta = \cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta$.

Exercise XXX. Page 137

5. $\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \dots$
6. $x \cot x = 1 - \frac{x^2}{3} - \frac{x^4}{45} - \frac{2x^6}{945} - \dots$
7. $\sin 10^\circ = 0.173648$; $\cos 10^\circ = 0.984808$.
8. $\tan 15^\circ = 0.267958$.



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